Absorption and amplification in a resonant medium

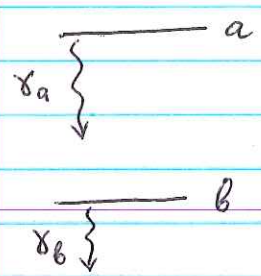
Wave equation describes the effect of an atom on e-m field

$$-\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\mu_0\underline{\frac{\partial^2\vec{P}}{\partial t^2}} \qquad \vec{P} = \wp_{ab}\,\rho_{ab}$$

slowly-varying amplitude/phase approximation

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = \underline{\frac{ik}{2\varepsilon_0}P_0} \qquad P_0 = 2\tfrac{1}{2}\wp_{ab}\,\rho_{ab}$$

Bloch equations (density matrix equations) describe the effect of e-m field(s) on an atomic quantum state



$$\dot{\rho}_{aa} = -\gamma_a\rho_{aa} + i\frac{\wp_{ab}E_0}{2\hbar}(\rho_{ab}-\rho_{ba})$$

$$\dot{\rho}_{bb} = \underbrace{-\gamma_b\rho_{bb}}_{\text{loss}}\ \underbrace{+\gamma_a\rho_{aa}}_{\text{repumping}} - i\frac{\wp_{ab}E_0}{2\hbar}(\rho_{ab}-\rho_{ba})$$

$$\dot{\rho}_{ab} = -(\gamma_{ab}+i\Delta)\rho_{ab} - i\frac{\wp_{ab}E_0}{2\hbar}(\rho_{aa}-\rho_{bb})$$

$\Delta = \omega - \omega_{ab}$ $\qquad \gamma_{ab} = \frac{1}{2}(\gamma_a+\gamma_b)$ (if no additional decoherence)

These equations are for two-level system, in a rotating wave approximation

In many situations we analyse the channels for population decay or pumping and decoherence, and add corresponding terms into the Bloch equations

If a level is stable (ground state) $\gamma_i = 0$

Causes of population decay:

- spontaneous emission
- state-changing collisions
- limited interaction times

All these processes affect the coherences as well, but there may be additional sources of decoherence (with no population change)

- collisions
- inhomogeneities in the system /interaction

Typically, two relaxation times are assigned to a ~~two population~~ / an atomic ~~state~~ transition

T_1 – $1/e$ decay time of the population difference
T_2 – ———— " ———— of the induced dipole (coherence)
→ originates from NMR

For pure radiative decay $T_1 \simeq \frac{1}{2}\left(\frac{1}{\gamma_a} + \frac{1}{\gamma_b}\right)$

$$T_2 = \frac{1}{\gamma_{ab}}$$

in case of additional dephasing γ_{dph}

$$T_2 = \frac{1}{\gamma_{ab} + \gamma_{dph}}$$

It is also common to introduse T_2^* – decoherence due to inhomogeneity within an ensemble of "oscillators"; it appears in the formal derivations during taking an ensemble averaging.

Coming back to our two-level system

$$\dot{\rho}_{ab} = -(\gamma_{ab} + i\Delta)\rho_{ab} + \frac{i}{2}\Omega(\rho_{aa} - \rho_{bb}) \qquad \Omega = \frac{\wp_{ab} E_0}{\hbar}$$

Rabi frequency

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a \rho_{aa} - \frac{1}{2} i\Omega(\rho_{ba} - \rho_{ab})$$

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b \rho_{bb} + \frac{i}{2}\Omega(\rho_{ba} - \rho_{ab})$$

λ_a, λ_b – repumping terms
$\gamma_a, \gamma_b, \gamma_{ab}$ – relaxation rates

Steady-state approximation $\frac{d\rho_{ij}}{dt} = 0$

$$\rho_{ab} = \frac{i\Omega}{2} \frac{\rho_{aa} - \rho_{bb}}{\gamma_{ab} + i\Delta} \qquad \rho_{ba} = \rho_{ab}^*$$

$$\rho_{ba} - \rho_{ab} = \frac{\Omega}{2}(\rho_{aa} - \rho_{bb})\left[-\frac{i}{\gamma_{ab} - i\Delta} - \frac{i}{\gamma_{ab} + i\Delta}\right] =$$

$$= -i\Omega(\rho_{aa} - \rho_{bb})\frac{\gamma_{ab}}{\gamma_{ab}^2 + \Delta^2}$$

and $\frac{i}{2}\Omega(\rho_{ba} - \rho_{ab}) = \frac{\Omega^2}{2}\frac{\gamma_{ab}}{\gamma_{ab}^2 + \Delta^2}(\rho_{aa} - \rho_{bb})$

$$\rho_{aa} = \frac{1}{\gamma_a}\left(\lambda_a - \frac{i\Omega}{2}(\rho_{ba} - \rho_{ab})\right)$$

$$\rho_{bb} = \frac{1}{\gamma_b}\left(\lambda_b + \frac{i\Omega}{2}(\rho_{bb} - \rho_{ab})\right)$$

Without the applied e-m field

$$\rho_{aa} - \rho_{bb} = \lambda_a/\gamma_a - \lambda_b/\gamma_b = N$$

unsaturated population differences

With the field

$$\rho_{aa}-\rho_{bb} = N - \frac{\Omega^2}{2}\left(\frac{1}{\gamma_a}+\frac{1}{\gamma_b}\right)\frac{\gamma_{ab}}{\gamma_{ab}^2+\Delta^2}(\rho_{aa}-\rho_{bb})$$

$$T_1 = \frac{1}{2}\left(\frac{1}{\gamma_a}+\frac{1}{\gamma_b}\right) \qquad \mathcal{L}(\Delta) = \frac{\gamma_{ab}^2}{\gamma_{ab}^2+\Delta^2}$$ Lorentzian function (normalized)

$$\rho_{aa}-\rho_{bb} = N - \Omega^2 T_1 T_2\, \mathcal{L}(\Delta)(\rho_{aa}-\rho_{bb})$$

$$\rho_{aa}-\rho_{bb} = \frac{N}{1+\underbrace{\Omega^2 T_1 T_2\, \mathcal{L}(\Delta)}_{\text{saturation}}}$$ spectral transition profile

The stronger is the field, the smaller is the population difference

Saturation intensity: $\overbrace{\Omega^2 T_1 T_2}^{I} = 1 \Rightarrow I_s = \frac{c\varepsilon_0(\hbar/\wp_{ab})^2}{T_1 T_2}$

$$\rho_{ab} = \frac{i\Omega}{2}\frac{1}{\gamma_{ab}+i\Delta}\cdot\frac{N}{1+I\cdot\mathcal{L}(\Delta)}$$

Induced polarization

$$P_0(z) = -i\frac{\wp_{ab}^2}{\hbar}\frac{N}{1+I\,\mathcal{L}(\Delta)}\frac{1}{\gamma_{ab}+i\Delta}\mathcal{E}(z)$$

In general $P_0 = \varepsilon_0 \chi \mathcal{E}$

$$\chi(\mathcal{E}) = -i\frac{\wp_{ab}^2}{\varepsilon_0\hbar}\frac{N}{1+I(\mathcal{E})\mathcal{L}(\Delta)}\frac{1}{\gamma_{ab}+i\Delta}$$

$$\chi(\mathcal{E}) = \frac{-\Delta - i\gamma_{ab}}{\Delta^2+\gamma_{ab}^2}\frac{\wp_{ab}^2}{\varepsilon_0\hbar}\frac{N}{1+I(\mathcal{E})\cdot\mathcal{L}(\Delta)}$$

The absorption coefficient

$$\alpha = +\frac{k}{2}\chi'' = -\frac{\wp_{ab}^2 N}{\varepsilon_0\hbar\gamma_{ab}}\frac{k}{2}\frac{\mathcal{L}(\Delta)}{1+I(\mathcal{E})\mathcal{L}(\Delta)} = \alpha_0\frac{\mathcal{L}(\Delta)}{1+I(\mathcal{E})\mathcal{L}(\Delta)}$$

Very weak e-m field $I \ll 1$

$$\alpha(\Delta) = \alpha_0 \mathcal{L}_1(\Delta) = \alpha_0 \frac{\gamma_{ab}^2}{\gamma_{ab}^2 + \Delta^2}$$ same lineshape as classical case

α_0 resonant unsaturated absorption

$$\alpha_0 = -\frac{k}{2} \frac{\wp_{ab}^2 N}{\epsilon_0 \hbar \gamma_{ab}}$$ $N = \rho_{aa}^{(0)} - \rho_{bb}^{(0)}$ with no e-m field

$\alpha_0 > 0$ $N < 1$ $\rho_{aa}^{(0)} < \rho_{bb}^{(0)}$ no population inversion

$\lambda_a/\gamma_a < \lambda_b/\gamma_b$ system is absorbing

$N > 1$ $\rho_{aa}^{(0)} > \rho_{bb}^{(0)}$ population inversion

$\lambda_a/\gamma_a > \lambda_b/\gamma_b$

$\alpha_0 > 0$ $$\frac{\partial \mathcal{E}}{\partial z} = -\alpha_0 \mathcal{E} = |\alpha_0| \mathcal{E}$$

$$\mathcal{E} = \mathcal{E}(z=0)\, e^{|\alpha_0| z}$$

gain medium

Population inversion is an essential condition for amplification

For a stronger e-m field

$$\frac{\partial \mathcal{E}}{\partial z} = -\alpha \mathcal{E}$$

Dimentionless intensity $I = \frac{\wp_{ab}^2 \mathcal{E}^2}{\hbar^2} T_1 T_2 = \frac{\text{real intensity}}{I_s}$

$$\frac{dI}{dz} = -2\alpha(\Delta) I = -2\alpha_0 I \underbrace{\frac{\mathcal{L}_1(\Delta)}{1 + I \mathcal{L}_1(\Delta)}}_{\text{saturation}}$$

Higher $I \rightarrow$ less absorption

Solving this equation:

$$\ln\left(\frac{I(z)}{I(0)}\right) + \left(I(z) - I(0)\right)\cancel{\alpha_1} = -2\alpha_0\alpha_1 z$$

no nice-looking general formula

$I \ll 1$

$$I(z) = I(0)\, e^{-2\alpha_0 z \alpha_1}$$

$I \gg 1$ $\qquad I(z) = I(0) - 2\alpha_0 z$