**PHYS 404/690 Quantum and Nonlinear Optics**

***Problem set # 6*** *(due March, 20)*

Each problem is 10 points. The problems marked with $*$ are required for graduate students only, and are extra credit problems for undergraduates.

**P1** Prove that coherent states are not orthogonal, *i.e.* that
$\langle\alpha|\beta\rangle = exp\{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta)\}$.

**P2** A single photon source is a necessary component of many quantum cryptography devices. Due to the lack of commercially available true single photon sources, many experiments use a strongly attenuated laser pulses with average photon number per pulse much less than zero. This is not optimal solution, since there is always a non-zero probability for having a pair of photons, which is a security risk. Assume that for a particular protocol sets a limit $p \ll 1$ to the ratio of two-photon pulses with respect to the singe-photon pulses. What is the average number of photons in this attenuated coherent state?

**P3** Verify that the quantum fluctuations of the field quadrature operators $\hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$ are the same as for the vacuum when the field is in a coherent state.

**P4** Calculate the output state after the beam splitter when a two-photon state $|2\rangle$ is sent to one of its inputs.

**P5*** Using Baker-Hausdorf lemma (below) prove that the unitary operator $\hat{U} = exp\left[i\frac{\pi}{4}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger)\right]$ describes the transformation between input and output modes of a 50/50 beam splitter. Now consider the operator $\hat{U}_\theta = exp\left[i\frac{\theta}{2}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger)\right]$, which corresponds to the more general case where the beam splitter is *not* 50/50. Obtain the corresponding transformation of the mode operators and relate the angle $\theta$ with the transmission/reflection parameters $r, t, r'$ and $t'$.
*Baker-Hausdorf lemma:*
$$e^{iG\lambda}Ae^{-iG\lambda} = A + i\lambda[G, A] + \frac{(i\lambda)^2}{2!}[G, [G, A]] + \ldots + \frac{(i\lambda)^n}{n!}\underbrace{[G, [G, [G, \ldots [G}_{n\ times}, A]]]\ldots] + \ldots$$