Identical particles

If particles are indistinguishable, that must be reflected in their collective wave function

Bosons (~~even~~ integer spin particles)
Wave function is symmetric to permutations

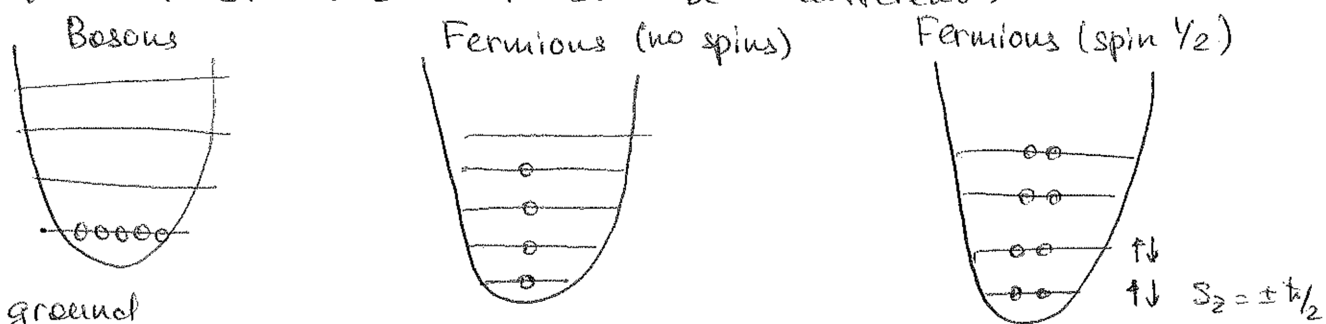
$$\psi(\ldots \vec{r}_A \ldots \vec{r}_B \ldots) = \psi(\ldots \vec{r}_B \ldots \vec{r}_A \ldots)$$

or, more generally, $\psi(A,B) = \psi(B,A)$

Fermions (half-integer spin particles)
Wave function is anti-symmetric under permutations

$$\psi(A,B) = -\psi(B,A)$$

Consequence: Pauli exclusion principle
No two fermions can occupy ~~the~~ a quantum state with fully identical set of quantum numbers (at least one must be different)



ground state

if no information about spin is given (~~likely~~ realistically they likely are in the same spin state)

each state is now doubly degenerate

Fermi energy – ~~highest~~ energy of the most energetic particle / highest occupied energy level in the fermionic collective ground state.

Free fermions: each particle occupies a "bubble", in a momentum space

For a given kinetic energy $E = \frac{\hbar^2 k^2}{2m}$

$$\Delta k_x \cdot \Delta k_y \cdot \Delta k_z = \frac{\pi^3}{V}$$ (or $\frac{1}{2}$, if spin is taken into account)

For N fermions

$$N \cdot \frac{\pi^3}{2V} = \frac{1}{8} \cdot \frac{4\pi}{3} k_f^3 \Rightarrow k_f = \sqrt[3]{3\pi^2 N/V}$$

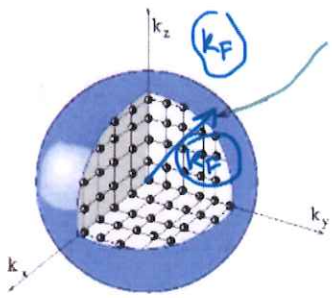
$$E_f = \frac{\hbar^2}{2k} k_f^2 = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

Free electron gas (ground state energy)

N

Fermi sphere

(all occupied states)



Fermi surface

(last occupied states)

k_F (Fermi wave vector)

↓

radius of the Fermi sphere

$\varepsilon_F = \frac{k_F^2}{2}$, Fermi energy

(The highest occupied energy)

Perturbation theory, or how to avoid solving Schrodinger equation (more than once)

Idea: we can fully characterize a quantum systenee $\hat{H}_0$, $\{|\psi_n^{(0)}\rangle\}$, $\{E_n^{(0)}\}$

Then we introduce a small disturbance (perturbation) to this system $\hat{V}$ or $\hat{V}(t)$
Then we will continue to use the original (unperturb) basis $\{\psi_n^{(0)}\}$ to find ~~___~~ solutions for the perturbed Hamiltonian $\hat{H}+\hat{V}$

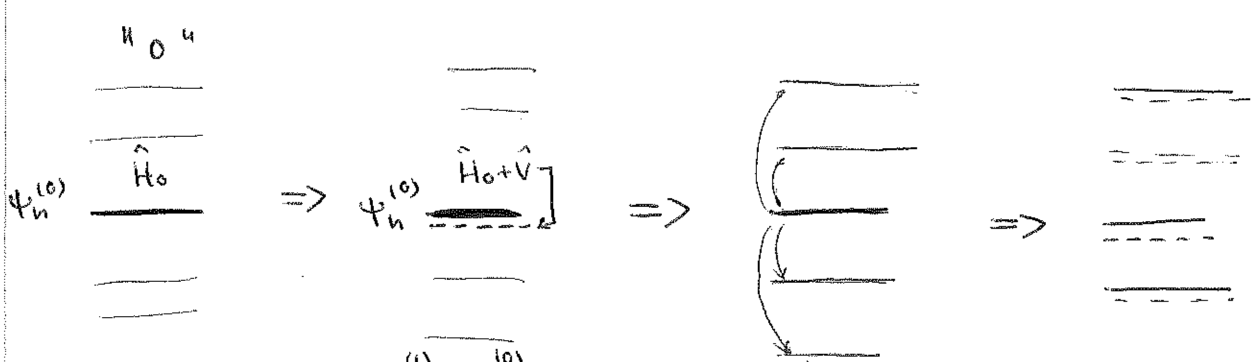
Time-independent PT

$|\psi\rangle = \sum_n c_n |\psi_n\rangle \qquad (\hat{H}+\hat{V})|\psi\rangle = E|\psi\rangle$

Then we assume that V is small
$\langle\hat{V}\rangle \ll \Delta E^{(0)}$ (where $\Delta E^{(0)}$ is a characteristic energy scale of the unperturbed system)
Thus we are looking for the corrections as a power series in V_{nm}

$$E_n^{(1)} = \langle\psi_n^{(0)}|\hat{V}|\psi_n^{(0)}\rangle \qquad \psi_n^{(1)} = \sum_{m\neq n} c_{mn}^{(1)}\psi_m^{(0)}$$

$$E_n^{(2)} = \sum_{m\neq n}\frac{|\langle\psi_m^{(0)}|\hat{V}|\psi_n^{(0)}\rangle|^2}{E_n^{(0)}-E_m^{(0)}} \qquad c_{mn}^{(1)} = \frac{\langle\psi_m^{(0)}|V_{mn}|\psi_n^{(0)}\rangle}{E_n^{(0)}-E_m^{(0)}}$$



$E_n^{(1)} = E_n^{(0)}+\langle V\rangle \qquad \psi_n^{(0)}+\sum_{m\neq n} c_{mn}\psi_m^{(0)}$

each energy level is now affected by $\hat{V}$

Degenerate states: no difinitive basis

—— s-fold degenerate state for $E_n^{(0)} = \{\psi_s\}$

any combination of ψ_s is an eigenstate

Then we specify the basis to use by choosing the one that is also an eigenbasis of $\hat{V}$ $\Rightarrow$ $\{\psi_s^{(new)}\}$ $\quad \hat{V}\psi_s^{(new)} = V_s \psi_s^{(new)}$

$$\Delta E_s^{(1)} = \langle \psi_s^{(new)} | \hat{V} | \psi_s^{(new)} \rangle = V_s$$

Why we need to do that? To keep our assumption that the perturbation only <u>weakly</u> affect our original system

Time-dependent perturbation

$$|\psi(t)\rangle = \sum_n c_n(t)\, \psi_n^{(0)}(t) = \sum_n c_n(t) |\psi_n\rangle\, e^{-iE_n t/\hbar}$$

must be solutions of time-dependent Schrodinger eq

$$i\hbar \frac{\partial \psi}{\partial t} = (\hat{H}_0 + \hat{V}(t))\, \psi(t)$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \sum_k c_k(t=0) \int_0^t \langle \psi_n^{(0)} | \hat{V}(t') | \psi_k^{(0)} \rangle\, e^{i\omega_{nk}t'}\, dt$$

(or start time) $\qquad \omega_{nk} = \dfrac{E_n - E_k}{\hbar}$

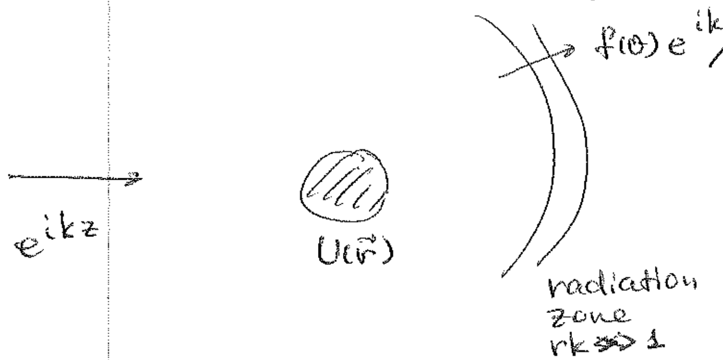
if at $t=0$ $\quad c_n = \delta_{ni}$ (defined state)

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_n^{(0)} | \hat{V}(t') | \psi_i^{(0)} \rangle\, e^{i\omega_{ni}t'}\, dt'$$

Scattering (elastic)

Born approximation = perturbation theory

General solution form of the Schrodinger eqn for scattering



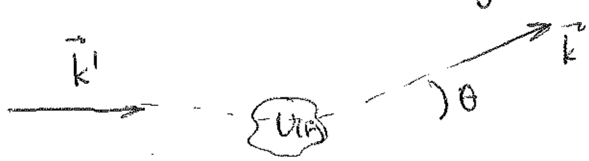
$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \times$$

$$\times\, U(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3\vec{r}_0$$

Approximation $U(\vec{r})$ has weak effect on $\psi(\vec{r})$

$$\cancel{\psi(\vec{r}) = A e^{ikz} - \frac{m}{2\pi\hbar^2}}$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}'-\vec{k})\vec{r}_0}\, U(\vec{r}_0)\, d^3\vec{r}_0$$



$\vec{k}' = k\vec{e}_z$

$\vec{k} = k\vec{e}_r$

$k = \sqrt{\frac{2mE}{\hbar^2}}$

Partial wave analysis (for a spherically-symmetric potential)

No change in an angular momentum as a result of scattering $\rightarrow$ we can consider each l wave separately ($m=0$)

$$\psi(r,\theta) = A\left\{e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)\, P_l(\cos\theta)\right\}$$

in the intermediate region ($V(\vec{r})=0$, but $kr \sim 1$)

Radiation zone ($kr \gg 1$)

$$\psi(r,\theta) = A\left\{e^{ikr} + \underbrace{\sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta)}_{f(\theta)}\, e^{ikr}/r\right\}$$

~~ind~~ scattering region

solving SE for $V(\vec{r}) \neq 0$, find a_l from boundary conditions

One can express a_l through a phase

$$a_l = e^{i\delta_l}\,\frac{\sin\delta_l}{k}$$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\,\frac{\sin\delta_l}{k}$$

$$\sigma = \sum \sigma_l = 4\pi\sum(2l+1)\,|a_l|^2 = \sum_{l=0}^{\infty} 4\pi\,(2l+1)\,\frac{\sin^2\delta_l}{k^2}$$