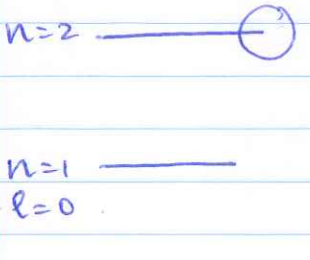
Hyperfine interaction

Where we are with H-atom level structure

	Coulumb interaction	+	Fine Structure
$n=$		:	

Good quantum numbers : $n\ l\ s\ j\ m_j$

i.e each state can be fully define but a unique combination of these quantum numbers



Hyperfine interaction arises from interaction of the electron and proton (or nucleus, in general) magnetic moments.

Electron magnetic moment $\vec{\mu}_e = -\mu_B (\vec{L} + 2\vec{S})\vec{B}$ $\mu_B = \frac{e\hbar}{2m_e c}$

Proton magnetic moment $\vec{\mu}_p = g_p \mu_N \vec{I} \cdot \vec{B}$ $\mu_N = \frac{e\hbar}{2m_p c}$

$g_p = 5.58$

It is possible to derive a full expression for a proper hyperfine interaction hamiltonian

$$\hat{H}'_{hfs} = \left[-\frac{8\pi}{3} \vec{\mu}_p \vec{\mu}_s \delta^3(\vec{r}) \right]_{l=0} + \left[\frac{\vec{\mu}_p \vec{\mu}_s}{r^3} - \frac{3(\vec{\mu}_p \vec{r})(\vec{\mu}_s \vec{r})}{r^5} - \frac{\vec{\mu}_p \vec{\mu}_e}{r^3} \right]_{l>0}$$

To simplify our life, we are going to restrict ourselves with $l=0$ case, so that $\vec{S} = \vec{J}$

In this case, one can show that the first-order correction is given by

$$\langle \hat{H}'_{Hfs} \rangle_{\ell=0} = \frac{a_n}{\hbar^2} \langle \hat{\vec{I}} \cdot \hat{\vec{S}} \rangle \qquad \text{where } a_n = \frac{2}{3n^3} 2 g_p \alpha^2 \frac{m_e}{m_p}$$

and thus we can actually approximate the hyperfine interaction hamiltonian for $\ell=0$ states as

$$\hat{H}'_{Hfs} = \frac{a_n}{\hbar^2} \hat{\vec{I}} \cdot \hat{\vec{S}}$$

$\vec{I}$ – nuclea (proton) spin
$\vec{S}$ – electron spin

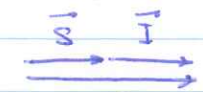
So the states we are considering have many quantum numbers already fixed:
n (affects only the value of a), $\ell=0$, $s=j=1/2$
$I=1/2$, so the only independent "variables" are $m_s = \pm 1/2$ and $m_I = \pm 1/2$.
Thus, the state is 4x degenerate, and to calculate the first-orde perturbation due to hypefine interaction, we need to identify the basis in which this perturbation is diagonal. Following the same procedure as we ~~dit~~ use for the spin-orbit interaction, we take an educated guess that ~~the~~ the basis of total atomic spin $\vec{F} = \vec{J} + \vec{I} = \vec{S} + \vec{I}$ (for $\ell=0$ states) will do the trick

$$|m_s\, m_I\rangle \quad \Rightarrow \quad |f, m_f\rangle$$

Proper spin addition: electron (spin 1/2), proton (spin 1/2)

Two spins are either aligned or anti aligned

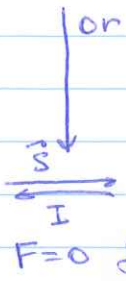
$\vec{F} = \vec{S} + \vec{I}$ ⟶ either



$\hat{F}^2 |f m_f\rangle = \hbar^2 f(f+1) |f m_f\rangle$

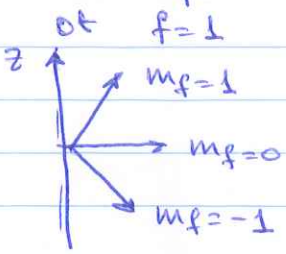
$f = 1$

or



$F = 0$ $f = 0$

Three possible orientations of $f = 1$



We can connect $|S, m_S\rangle$ and $|I, m_I\rangle$ states
$S = I = 1/2$ $m_S, m_I = \pm 1/2$ (or $|\uparrow\rangle, |\downarrow\rangle$)

$f = 1$ $m_f = 1$ $\Rightarrow$ $m_S = m_I = 1/2$

$m_f = -1$ $\Rightarrow$ $m_S = m_I = -1/2$

but $m_f = 0$ $\Rightarrow$ $\frac{1}{\sqrt{2}} (|\uparrow_S \downarrow_I\rangle + |\downarrow_S \uparrow_I\rangle)$ symmetric combination

$f = 0$ $m_f = 0$ $\Rightarrow$ $\frac{1}{2} (|\uparrow_S \downarrow_I\rangle - |\downarrow_S \uparrow_I\rangle)$ anti-symmetric combination

(in general, we would use Klepsh – Gordon coefficients to figure out the connection b/w different bases).

The hyperfine hamiltonian is not diagonal in $|m_s, m_I\rangle$ basis:

$$\hat{\vec{S}}\cdot\hat{\vec{I}} = \hat{S}_x\hat{I}_x + \hat{S}_y\hat{I}_y + \hat{S}_z\hat{I}_y = \frac{1}{2}(\hat{I}_+\hat{S}_- + \hat{I}_-\hat{S}_+) + \hat{S}_z\hat{I}_z$$

and since $\hat{S}_+|m_S=1/2, m_I\rangle = 0$ $\quad \hat{S}_+|m_S=-1/2, m_I\rangle = \frac{\hbar}{2}|m_S=1/2, m_I\rangle$

$\hat{S}_-|m_S=1/2, m_I\rangle = \frac{\hbar}{2}|m_S=-1/2, m_S\rangle$ $\quad \hat{S}_-|m_S=-1/2, m_I\rangle = 0$

(and similar for $I_\pm$)

the first two terms give non-zero off-diagonal elements.

On the other hand, since $\hat{\vec{F}} = \hat{\vec{S}} + \hat{\vec{I}}$ and

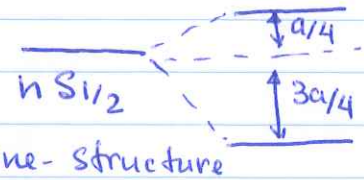
$\hat{F}^2 = \hat{S}^2 + \hat{I}^2 + 2\hat{\vec{S}}\hat{\vec{I}}$ $\Rightarrow$ $\hat{\vec{S}}\cdot\hat{\vec{I}} = \frac{1}{2}(\hat{F}^2 - \hat{I}^2 - \hat{S}^2)$

$$\langle f m_f|\, \hat{\vec{S}}\cdot\hat{\vec{I}}\, |f m_f\rangle = \frac{1}{2}\left[f(f+1) - I(I+1) - S(S+1)\right] =$$

$$= \frac{1}{2}\left[f(f+1) - 3/2\right]\hbar^2$$

For $f=1 \Rightarrow$ $\langle H'_{Hfs}\rangle_{f=1} = \frac{a}{2}\left[f(f+1) - 3/2\right] = a/4$

$f=0$ $\quad \langle H'_{Hfs}\rangle_{f=0} = -3a/4$



$f=1$ (triple degenerate $m_f = 0, \pm 1$)

$f=0$ (singlet, $m_f = 0$)

For the hydrogen ground state $a = 1.4204\ GHz$ ($\lambda = 21cm$)
Hydrogen masers operate on that wavelength.

Similar effect for any other nucleus: a value changes for each state and element, but the structure stays the same

^{87}Rb : $5S_{1/2}$, $I = 3/2$ $\Rightarrow$ Two hyperfine states $F = 1, 2$

^{85}Rb : $5S_{1/2}$, $I = 5/2$ $\Rightarrow$ ——— " ——— $F = 2, 3$

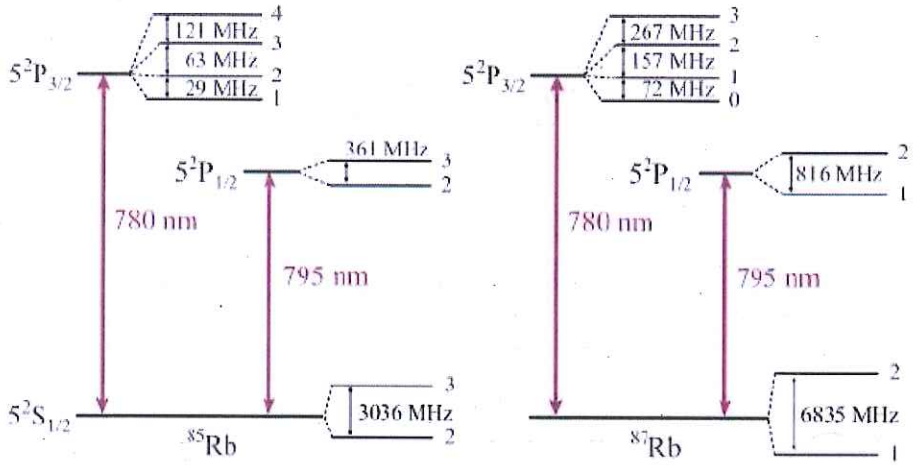
Moreover, the proper calculations for $\ell \neq 0$ states shows very similar dependense, with S replaced with J

$$\langle H'_{Hfs} \rangle = \langle \psi_{n\ell sjIfm_f} | H'_{Hfs} | \psi_{n\ell sjIfm_f} \rangle = \left[\frac{\alpha^2 m_e/m_p g_p}{2n^3(2\ell+1)} \right] \times$$

$$\times \frac{f(f+1) - I(I+1) - j(j+1)}{j(j+1)}$$

For example for $5P_{1/2}$ state in Rb ($J = 1/2$, $I = 3/2$) possible $f = 1$ or 2 => two hyperfine levers

but for $5P_{3/2}$ state - $J = 3/2$, $I = 3/2$ there are 4 possible states $f = 0, 1, 2, 3$ so the hyperfine structure will have 4 possible levels.



Level notation: $n\,{}^{2s+1}\ell_j$

where: $\ell = 0 \rightarrow S$, $\ell = 1 \rightarrow P$, $\ell = 2 \rightarrow D$, $\ell = 3 \rightarrow F$;

$5^2P_{1/2}$: $n = 5$, $\ell = 1$, $S = 1/2$, $j = 1/2$