Brief reminder / summary of usage of Green's functions

1D case: ordinary nonlinear differential equation

$D_t[y] = f(t)$ where $D_t[..]$ is a differential operator, need to find $y(t)$ that is a solution

for example: driven oscillator: $\overbrace{y^2 + \omega^2 y}^{D_t[y]} = f(t)$

If one can find a Green's function $G(t,t')$ such that

$$D_t[G(t,t')] = \delta(t-t') \quad \left[\frac{d^2}{dt^2} G(t,t') + \omega_0^2 G(t,t') = \delta(t-t') \right]$$

Then the solution of the original equation is $y(t) = y_0(t) + \int_0^\infty G(t,t') f(t') dt'$

where $y_0(t)$ is a ~~homo~~ solution of a homogenious equation $D_t[y]=0$ $[y'' + \omega_0 y = 0]$

Same idea works in a 3D case — in fact you probably used it!

Electrostatic potential for a known charge distribution $\nabla^2 \varphi = \rho(\vec{r})$

Point charge at $\vec{r}_0 \longrightarrow d\rho = q \cdot \delta(\vec{r}-\vec{r}_0)$

$\nabla^2 \varphi_0(\vec{r},\vec{r}_0) = q \cdot \delta(\vec{r}-\vec{r}_0)$ Green's function equation

$$\varphi_0 = \frac{q}{|\vec{r}-\vec{r}_0|}$$

The solution of the original equation

$$\varphi(\vec{r}) = \sum_V \varphi_0(\vec{r},\vec{r}_0) = \int_V \frac{dq}{|\vec{r}-\vec{r}_0|} = \underbrace{\int_V \frac{\rho(\vec{r}_0)}{|\vec{r}-\vec{r}_0|} d^3\vec{r}_0}_{\int G(\vec{r},\vec{r}_0)\rho(\vec{r}_0) d^3\vec{r}_0}$$

Green's function solution