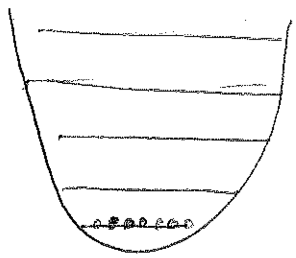
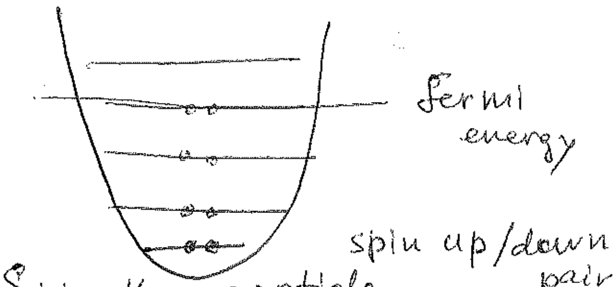
Fermionic gas

So far, we considered the behavior of identical particles ~~with~~ in bound states





Spin 0 particles
ground state - all particles are in the same lowest energy state

Spin 1/2 particle
ground state - all low energy states are filled up to the Fermi energy

For bound fermions the # of particles per energy state equal the degeneracy of this state, as at least one quantum number must be different

What about unbound fermions?
No quantization → no descrete states → no Pauli exclusion principle? Not so fast

Since uncertainty principle does not allow us to know both momentum and position exactly, and no two particles can have ~~exact~~ same position and momentum, each particle has its own private "bubble" $\Delta p \cdot \Delta x \sim \hbar$

Examples of (almost) free and (almost) non-interacting gas of fermions

- neutron star
- electrons inside a solid

Free particle (no forces, no potential energy)

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

Eigenstates: $\varphi(\vec{r}) = N \sin(k_x \cdot x + \varphi_x) \sin(k_y \cdot y + \varphi_y) \sin(k_z \cdot z + \varphi_z)$

Eighen energy $E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$

For a truly free particles k_x, k_y, k_z - any values

However, if there are <u>some</u> restrictions, these wave vectors will be quantized

We approximate the boundary of a solid as

$$U(x,y,z) = \begin{cases} 0 & 0 < |x,y,z|^2 < L_x, L_y, L_z \\ \infty & \text{otherwise} \end{cases}$$

$$\varphi(\vec{r}) = \sqrt{\frac{8}{L_x L_y L_z}} \sin\frac{\pi n x}{L_x} \sin\frac{\pi m y}{L_y} \sin\frac{\pi s z}{L_z}$$

$$k_x = \frac{\pi n}{L_x}, \quad k_y = \frac{\pi m}{L_y}, \quad k_z = \frac{\pi s}{L_z}, \quad E_{nms} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{s^2}{L_z^2}\right)$$

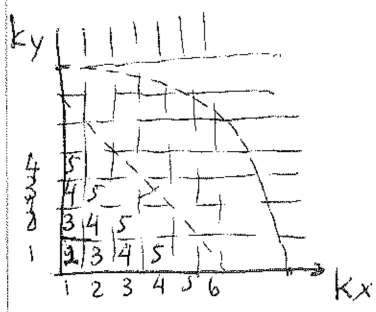
Electrons are delocalized (their positions are not known), but each k-vector can only change by the same amount

$$\Delta k_x = \frac{\pi}{L_x} \quad \Delta k_y = \frac{\pi}{L_y} \quad \Delta k_z = \frac{\pi}{L_z}$$

So in reality each particle occupies a specific "block" in k-vector space

$\Delta \vec{k} = \Delta k_x \cdot \Delta k_y \cdot \Delta k_z = \pi^2 / L_x L_y L_z$

For spin 1/2 particles only 2 particles can occupy the same state with n, m, s



Same energy lines $k_x^2 + k_y^2$ – a ~~q~~ quarter of a circle

($n+m$ is shown)

In 3D if we start filling up the availabe states from the lowest one ($n=m=s=1$), we will be layer by layer adding ~~layers~~ spherical shells such that $k_x^2 + k_y^2 + k_z^2 = \text{const}$

Assuming L_x, L_y, L_z very large $\pi^2 / L_x L_y L_z = \pi^2 / V$ – very small, so we can treat the steps as almost continuous

~~How many par~~ So if we have N particles, what is the highest energy "shell" they will reach? $E_f = \hbar^2 k_f / 2m$

"Volume" of the available space in k-space

$$= \frac{1}{8} \cdot \frac{4\pi}{3} k_f^3$$

Unit volume per particle $\frac{1}{2} \cdot \pi^3 / V$

$$N = \frac{1}{8}\frac{4\pi}{3} k_f^3 \Big/ \frac{1}{2}\pi^3 / V \qquad N = V \cdot \frac{k_f^3}{3\pi^2}$$

Particle density $\rho = N/V \Rightarrow \rho = k_f^3 / 3\pi^2 \Rightarrow k_f = \sqrt[3]{3\pi^2 \rho}$

$$E_f = \frac{\hbar^2 k_f^2}{2m} = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$$

That tells us that for a given electron density, the highest electron ~~density~~ energy is E_f

Can we now produce a distribution of number of electron with a given energy E (or a wavevector value k)?

Such ~~electrons~~ electrons occupy a spherical shell, with its volume to be $dV = \frac{1}{8}\cdot 4\pi k^2 dk$ $= \frac{1}{2}\pi k^2 dk$

$$dn(k) = \underset{\text{spin degeneracy}}{2} \times \frac{dV}{\underbrace{(\pi^3/V)}_{\text{a volume per one electron}}} = 2\cdot\frac{V}{\pi^3}\,\frac{1}{2}\pi k^2 dk = \frac{V}{\pi^2}k^2 dk$$

~~Thus~~, ~~the~~ We can ~~also~~ now use this distribution to calculate various relevant parameters.

Total energy $E_{tot} = \int_0^{k_F} E(k)\,dn(k) = \int \frac{\hbar^2 k^2}{2m}\,dn(k) =$

$$= \frac{\hbar^2}{2m}\cdot\frac{V}{\pi^2}\int_0^{k_F} k^4 dk = \frac{\hbar^2}{2m}\,\frac{V}{\pi^2}\,\frac{k_F^5}{5} = \frac{\hbar^2}{10m}\,\frac{V}{\pi^2}(3\pi^2)^{5/3}\left(\frac{N}{V}\right)^{5/3} =$$

$$= \frac{\hbar^2}{2m}\,\frac{3^{5/3}\cdot\pi^{4/3}}{5}\,N\cdot\left(\frac{N}{V}\right)^{2/3} = \frac{\hbar^2}{2m}\,\frac{3}{5}N\left(3\frac{N}{V}\pi^2\right)^{2/3} = \frac{3}{5}N\cdot E_F$$

$$E_{tot} = N\cdot\left(\frac{3}{5}E_F\right)$$

Average energy per ~~nucleon~~ particle $\frac{E_{tot}}{N} = \frac{3}{5}E_F$

Quantum pressure $dE_{tot} = - dW = - PdV$

$$P = - \frac{dE_{tot}}{dV} = - \frac{\hbar^2}{2m} \frac{3}{5} N (3N\pi^2)^{2/3} \frac{d}{dV}\left(\frac{1}{V^{2/3}}\right) = \frac{2}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \left(\frac{N}{V}\right)^{5/3}$$

$$P = \frac{\hbar^2}{5m} (3\pi^2)^{2/3} \rho^{5/3}$$

Higher density ~~cr~~ creates higher pressure