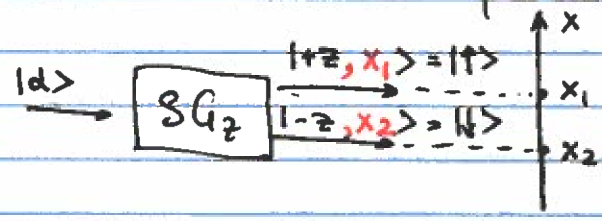
Position & momentum Operators

How do we measure a spin component of a particle



We actually measure the <u>position</u> of a particle, that is entangled with its spin

$$|\alpha\rangle \longrightarrow c_+ |+z, x_1\rangle + c_- |-z, x_2\rangle$$

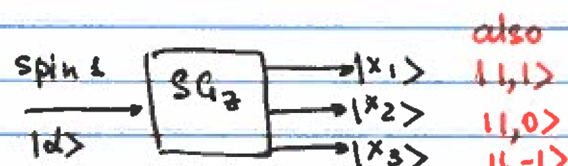
Actual measurement $\longrightarrow$ <u>position</u> $\longrightarrow$ implies $\longrightarrow$ <u>spin</u>

$\hat{x}|\uparrow\rangle = x_1|\uparrow\rangle \quad \Rightarrow$ we know that $\quad \hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$

$\hat{x}|\downarrow\rangle = x_2|\downarrow\rangle \quad \Rightarrow$ —— " —— $\quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

Eigenvalues of $\hat{x}|x\rangle = x|x\rangle$ — position of a particle is known to be x

Main difference from spin $\longrightarrow$ $\hat{x}$ has (or can have) continuous eigenvalues



$$|\alpha\rangle = c_1|x_1\rangle + c_2|x_2\rangle + c_3|x_3\rangle$$

reminder

$$|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3| = \hat{1}$$

$$\hat{1}|\alpha\rangle = |x_1\rangle\underbrace{\langle x_1|\alpha\rangle}_{c_1} + |x_2\rangle\underbrace{\langle x_2|\alpha\rangle}_{c_2} + |x_3\rangle\underbrace{\langle x_3|\alpha\rangle}_{c_3}$$

More output channels $\longrightarrow$ more $|x_i\rangle$ terms

$$c_i = \langle x_i|\alpha\rangle$$

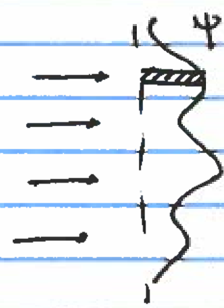
$P(x_1)$  $x_1, \Delta x$; $x_2, \Delta x$; $x_3, \Delta x$ $dP(x_1) = \rho(x_1)\,dx$

$$|\alpha\rangle = \sum_{i=1}^{N} |x_i\rangle\langle x_i|\alpha\rangle \qquad \Delta x \to 0$$

For infinitely large number of detectors

$$|\alpha\rangle = \int_{-\infty}^{+\infty} |x\rangle \underbrace{\langle x|\alpha\rangle}_{\text{function of } x}\, dx \qquad \hat{1} = \int_{-\infty}^{+\infty} |x\rangle\langle x|\, dx$$

Wave function: $\psi_\alpha(x) = \langle x|\alpha\rangle$ probability density



Probability to detect the particle b/w x and $x + dx$

$$dP(x) = |\psi(x)|^2 dx$$

Probability to find the particle b/w $x=a$ and $x=b$

$$P_{ab} = \int_a^b |\psi(x)|^2 dx \qquad \left(\text{similar to } P_{output} = |\langle out|\alpha\rangle| \right.$$

The same methodology still works

$$\langle S_z\rangle = \langle\alpha|\hat{S}_z|\alpha\rangle$$

$$\langle x\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \langle\alpha|\hat{x}\cdot\hat{1}|\alpha\rangle$$

$$= \int_{-\infty}^{+\infty} \langle\alpha|\underbrace{\hat{x}|x\rangle}_{x|x\rangle}\langle x|\alpha\rangle\, dx = \int_{-\infty}^{+\infty} x\cdot \underbrace{\langle\alpha|x\rangle}_{\psi^*(x)}\underbrace{\langle x|\alpha\rangle}_{\psi(x)}\, dx$$

$$= \int_{-\infty}^{+\infty} x\cdot P(x)\, dx \qquad P(x) = |\psi(x)|^2 \quad \text{probability density}$$

Momentum operator

Classical particle

$\bullet \cdots\cdots \bullet$

x $\qquad$ $x+dx = x + v_x \cdot dt$

$t=0$ $\qquad$ $t = dx$

if v_x is constant

$x \longrightarrow x+a$ $\qquad$ translation of a moving particle

$t=0$ $\qquad$ $t = v_x/a$

Quantum translation operator

$$\hat{T}(a) = e^{-i\hat{p}_x a/\hbar}$$

$\hat{p}_x$ – momentum operator

Eigenstates of the momentum operator describes the particle state with known momentum ~~p~~ value p (1-D case for now)

$$\hat{p}_x |p\rangle = p|p\rangle$$

Important : $\hat{x}$ and $\hat{p}_x$ do not commute!

$$[\hat{x}, \hat{p}_x] = i\hbar \quad \Rightarrow \quad \Delta x^2 \cdot \Delta p_x^2 \geq \hbar^2/4$$

~~Complete in~~

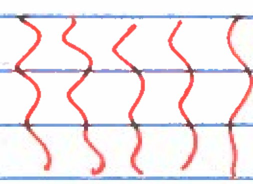
No uncertainty in position $\longrightarrow$ no info on momentum

localize particle

no way of knowing where it is going

No uncertainty in momentum $\longrightarrow$ no info on position

particle is completele delocolized



wave!

De Broigle wavelength $\lambda = \frac{2\pi\hbar}{p}$

States with known momentum ~~ha~~ represent waves with known wavelength

Just like for the spins, we can decompose the state of the particle in either x or p basis (or x- or p- representation)

$$|\alpha\rangle = \int_{-\infty}^{+\infty} \psi(x)\,|x\rangle\,dx \quad - \quad x\text{- representation}$$

or

$$|\alpha\rangle = \int_{-\infty}^{+\infty} \langle p|\alpha\rangle \cdot |p\rangle \cdot dp$$

P-representation is very useful when we ~~ta~~ use free particles where only there kinetic energy is measured.

$$\hat{K} = \frac{\hat{p}^2}{2m} \qquad \hat{K}\,|\hat{p}\rangle = \frac{1}{2m}\,\hat{p}^2\,|p\rangle = \underbrace{\frac{p^2}{2m}}_{\text{eigenvalue of } \hat{K}}\,|p\rangle$$

However, it is much more common to work with the position operator eigenstates. Especially if we have position-dependent potential energy

$\hat{p} = $ Momentum operator in x-basis

$$\hat{p} \longrightarrow -i\hbar\,\frac{\partial}{\partial x}$$

What it means: $|\beta\rangle = \hat{p}\,|\alpha\rangle$

Corresponding wave functions

$$\psi_\beta(x) = \langle x|\beta\rangle \qquad \psi_\alpha(x) = \langle x|\alpha\rangle$$

$\psi_\beta(x) = \langle x|\hat{p}|\alpha\rangle = \left[-i\hbar\frac{\partial}{\partial x}\right]\psi_\alpha(x)$ [proof in Townsend]

How does the state of the particle with known momentum looks like in x-basis? $\langle x|p\rangle = ?$

$$\hat{p}_x|p\rangle = p|p\rangle$$
$$\langle x|\hat{p}_x|p\rangle = p\langle x|p\rangle$$
$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle \implies \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

compare to a ~~constant~~ complex representation of a wave

$$f(x,t) = A\,e^{i2\pi x/\lambda + i2\pi t/T}$$

more familiar form → real part

$$f(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right)$$