To commute or not to commute?
That is uncertain

Definition: commutator of two operators
is $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = -[\hat{B},\hat{A}]$

Two operators commute $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$
for any $|\psi\rangle$
Then their order does not matter

Two operator does not commute if
$\hat{A}\hat{B}|\psi\rangle \neq \hat{B}\hat{A}|\psi\rangle$

The example of commuting operator:
$\hat{J}_z$ & $\hat{R}(\phi\hat{k}) = e^{-i\hat{J}_z\varphi/\hbar}$ $\qquad \hat{J}_z\hat{R} = \hat{R}\hat{J}_z$
(since $\hat{R}$ is written in terms of $\hat{J}_z$)
Easy to check in matix form

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & -e^{i\varphi/2} \end{pmatrix} = \cancel{\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e \end{pmatrix}}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Non-commuting operators — pauly mattices
(and spin components, and rotation ~~axe~~
matrices aroudd different orthogonal
axes)

Let's check that $\hat{\sigma}_x$ and $\hat{\sigma}_y$ don't commute

$$\hat{\sigma}_x \hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\hat{\sigma}_y \hat{\sigma}_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = \hat{\sigma}_x \hat{\sigma}_y - \hat{\sigma}_y \hat{\sigma}_x = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_z$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\sigma_z \; ; \; [\hat{\sigma}_y, \hat{\sigma}_z] = 2i\sigma_x \; ; \; [\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$$

Because $\hat{\sigma}_x, \hat{\sigma}_y$ & $\hat{\sigma}_z$ don't commute, they cannot be measured precisely at the same time.

Any two non-commuting (hermitian) operators are subject to the uncertainty principle

i.e. if $[\hat{A}, \hat{B}] = i\hat{C}$ then

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle \hat{C} \rangle|$$

where the uncertainty is defined as

$$(\Delta \hat{A})^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle \; ; \; (\Delta \hat{B})^2 = \langle (\hat{B} - \langle \hat{B} \rangle)^2 \rangle$$

(for proof see next page or textbook)

Mathematical proof of the uncertainty principle

Let's call $a = \langle \hat{A} \rangle$ expectation value of $\hat{A}$

$b = \langle \hat{B} \rangle$ ——— " ——— $\hat{B}$

For convenience let's define $|\psi_a\rangle = (\hat{A} - a)|\psi\rangle$

$|\psi_b\rangle = (\hat{B} - b)|\psi\rangle$

Since $\hat{A}, \hat{B}$ are hermitian operators $\hat{A}^{+} = \hat{A}$ then their ~~a~~ expectation values are real

$$\langle \psi_a | = \langle \psi | (\hat{A}^{+} - a^{*}) = \langle \psi | (\hat{A} - a)$$

$$\langle \psi_b | = \langle \psi | (\hat{B}^{+} - b^{*}) = \langle \psi | (\hat{B} - b)$$

Then $\Delta A^2 = \langle (\hat{A} - a)^2 \rangle = \overbrace{\langle \psi | (\hat{A} - a)}^{\langle \psi_a |} \overbrace{(\hat{A} - a) | \psi \rangle}^{| \psi_a \rangle} = \langle \psi_a | \psi_a \rangle$

$\Delta B^2 = \langle \psi_b | \psi_b \rangle$

Cauchy - Schwartz inequality

For any two states $\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$

similar to $|\vec{a}|^2 \cdot |\vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \cos\theta$

$$\Delta A^2 \cdot \Delta B^2 = \langle \psi_a | \psi_a \rangle \langle \psi_b | \psi_b \rangle \geq |\langle \psi_a | \psi_b \rangle|^2$$

$$|\langle \psi_a | \psi_b \rangle|^2 = (\mathrm{Re} \langle \psi_a | \psi_b \rangle)^2 + (\mathrm{Im} \langle \psi_a | \psi_b \rangle)^2 \geq (\mathrm{Im} \langle \psi_a | \psi_b \rangle)^2$$

$$\mathrm{Im} \langle \psi_a | \psi_b \rangle = \frac{\langle \psi_a | \psi_b \rangle - \langle \psi_b | \psi_a \rangle}{2i} = \frac{\langle \psi | (\hat{A} - a)(\hat{B} - b) | \psi \rangle}{2i} -$$

$$- \frac{\langle \psi | (\hat{B} - b)(\hat{A} - a) | \psi \rangle}{2i} = \frac{\langle \psi | \hat{A}\hat{B} - \hat{B}\hat{A} | \psi \rangle}{2i} = \frac{1}{2} \langle \psi | \hat{C} | \psi \rangle$$

$$\Delta A^2 \cdot \Delta B^2 \geq \frac{1}{4} |\langle \hat{C} \rangle|^2 \quad \text{or} \quad \Delta A \cdot \Delta B \geq \frac{1}{2} |\langle \hat{C} \rangle|$$

Components of spin vector don't commute

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$

$$[\hat{S}_x, \hat{S}_y] = \frac{\hbar^2}{4}[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\frac{\hbar^2}{4}\sigma_z = i\hbar\,\hat{S}_z$$

If all spins are aligned ; $|+z\rangle$ state

$$\langle +z|\hat{S}_z|+z\rangle = \frac{\hbar}{2}$$

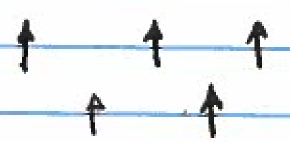
$$\Delta S_x \cdot \Delta S_y \geq \frac{\hbar}{2}|\langle \hat{S}_z\rangle| = \frac{\hbar^2}{4}$$

Guessing that $\Delta S_x = \Delta S_y$ due to symmetry

$$\Delta S_x = \Delta S_y \geq \hbar/2$$

This indicates that we cannot align and measure spins along the same precise orientation

Classical view



Quantum view