# A system of two spin-1/2 particles

① ②

$|\psi^{(1)}_{\pm}\rangle = |\underbrace{s_1 = 1/2}_{\text{constant}}, m_1 = \pm 1/2\rangle$ $\qquad$ $|\psi^{(2)}\rangle = |\underbrace{s_2 = 1/2}_{\text{constant}}, m_2 = \pm 1/2\rangle$

Two-particle states $\quad |\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle$
direct product

Two single-particle states are treated individually (not multiplied!)

$$|+z\rangle_{①} \otimes |+z\rangle_{②} \equiv |\overset{①}{+z}, \overset{②}{+z}\rangle$$

If an operator is defined for one particle, it will act only on that particle

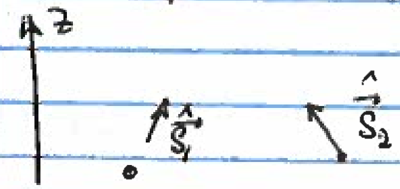
Since, for example, $\hat{S}_{1x}|+z\rangle_1 = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$
i.e. $\hat{S}_{1x}|+z\rangle_1 = \frac{\hbar}{2}|-z\rangle_1$

$$\hat{S}_{1x}|+z,+z\rangle = \left(\hat{S}_{1x}|+z\rangle_1\right) \otimes |+z\rangle_2 = \frac{\hbar}{2}|-z\rangle_1 \otimes |+z\rangle_2 = \frac{\hbar}{2}|-z,+z\rangle$$

$$\hat{S}_{2x}|+z,+z\rangle = |+z\rangle_1 \otimes \left(\hat{S}_{2x}|+z\rangle_2\right) = \frac{\hbar}{2}|+z,-z\rangle$$

$$\hat{S}_{1x}\hat{S}_{2x}|+z,+z\rangle = \left(\hat{S}_{1x}|+z\rangle_1\right) \otimes \left(\hat{S}_{2x}|+z\rangle_2\right) = \left(\frac{\hbar}{2}\right)^2 |-z,-z\rangle$$

1 Spin addition




vs.

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$

two individual spin-1/2 particles
state $\rightarrow$ $|m_1 = \pm 1/2, m_2 = \pm 1/2\rangle$
(defined in a common z-basis

total spin $\hat{\vec{S}}$

state $|s, m\rangle$ or $|j, m\rangle$

In classical physics addition of two equal vectors results in ~~any~~ a vector sum length having any ~~number~~ value b/w 0 and double of the original length.

In quantum angular momenta the resulting vectors can only have specific values
2 spin-1/2 $\longrightarrow$ spin = 1 or spin = 0

In general if $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$, then
$j = j_1+j_2, j_1+j_2-1 \ldots |j_1-j_2|$

For example $j_1 = 1/2, j_2 = 1/2 \rightarrow j = 1, 0$
$j_1 = 1, j_2 = 1 \longrightarrow j = 2, 1, 0$

Challenge $\rightarrow$ how to relate the basis of individual spins $|m_1, m_2\rangle$ to the basis of the total spin $|j, m\rangle$ ?

$$\hat{J}_1^2 |m_1, m_2\rangle = \hat{J}_1^2 |m_1\rangle \otimes |m_2\rangle = \frac{3\hbar^2}{4} |m_1\rangle \otimes |m_2\rangle = \frac{3\hbar^2}{4} |m_1, m_2\rangle$$

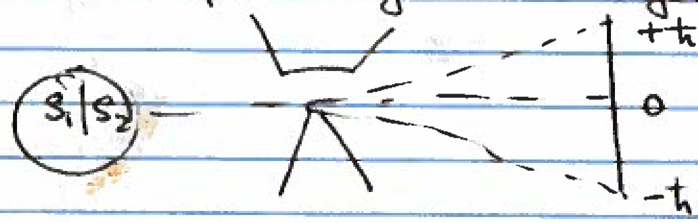
$$\hat{J}_{1z} |m_1, m_2\rangle = \hat{J}_{1z} |m_1\rangle \otimes |m_2\rangle = \frac{\hbar}{2} \cdot m_1 |m_1, m_2\rangle$$

same for $\hat{\vec{J}}_2$ operator

$\vec{J} = \vec{J}_1 + \vec{J}_2 \longrightarrow \hat{\vec{J}}_z = \hat{J}_{1z} + \hat{J}_{2z} \longrightarrow m = m_1 + m_2$

$\hat{J}^2 = (\hat{\vec{J}}_1 + \hat{\vec{J}}_2)^2 = \hat{\vec{J}}_1^{\,2} + \hat{\vec{J}}_2^{\,2} + 2\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2 = \hat{\vec{J}}_1^{\,2} + \hat{\vec{J}}_2^{\,2} +$
$+ 2\hat{J}_{1x}\hat{J}_{2x} + 2\hat{J}_{1y}\hat{J}_{2y} + 2\hat{J}_{1z}\hat{J}_{2}$

Remember our thought experiment about two spins "glued" together in a SG apparatus!



$|\uparrow\uparrow\rangle \longrightarrow m = 1, j = 1$

$|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$

$|\downarrow\downarrow\rangle \longrightarrow m = -1, j = 1$

$|1,+1\rangle \equiv |+1/2\rangle_1 \otimes |+1/2\rangle$

$\hat{J}_z |1,+1\rangle = (\hat{J}_{z_1} + \hat{J}_{z_2}) |+1/2\rangle \otimes |+1/2\rangle = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right) |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$

$\hat{J}_z |1,+1\rangle = \hbar |1,+1\rangle$

Similarly, $\hat{J}_z |1,-1\rangle = (\hat{J}_{z_1} + \hat{J}_{z_2}) |+1/2\rangle \otimes |+1/2\rangle = -\hbar |1,-1\rangle$

Tensor products of individual spins are eigen~~vecto~~ states of $\hat{J}_z$, but <u>not</u> $\hat{J}^2$

We can simplify our lives a little if we use $\hat{J}_\pm$ operators

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \Rightarrow \hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \quad \hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$

$\hat{J}_{x_1}\hat{J}_{x_2} = \frac{1}{4}(\hat{J}_{+1} + \hat{J}_{-1})(\hat{J}_{+2} + \hat{J}_{-2}) = \frac{1}{4}(\hat{J}_{+1}\hat{J}_{+2} + \hat{J}_{+1}\hat{J}_{-2} + \hat{J}_{-1}\hat{J}_{+2} + \hat{J}_{-1}\hat{J}_{-2})$

$\hat{J}_{y_1}\hat{J}_{y_2} = -\frac{1}{4}(\hat{J}_{+1} - \hat{J}_{-1})(\hat{J}_{+2} - \hat{J}_{-2}) = -\frac{1}{4}(\hat{J}_{+1}\hat{J}_{+2} + \hat{J}_{+1}\hat{J}_{-2} - \hat{J}_{-1}\hat{J}_{+2} + \hat{J}_{-1}\hat{J}_{-2})$

$\hat{J}_{x_1}\hat{J}_{x_2} + \hat{J}_{y_1}\hat{J}_{y_2} = \frac{1}{2}\hat{J}_{+1}\hat{J}_{-2} + \frac{1}{2}\hat{J}_{-1}\hat{J}_{+2}$

$\hat{J}_+ |j,m\rangle = \hbar$ [illegible] $|j,m+1\rangle \qquad \hat{J}_- |j,m\rangle = \hbar$ [illegible] $|j,m-1\rangle$

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_{+1}\hat{J}_{-2} + J_{-1}\hat{J}_{+2} + 2\hat{J}_{z_1}\hat{J}_{z_2}$$

State $|\uparrow_1\uparrow_2\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle$

$$\hat{J}^2|\uparrow_1\uparrow_2\rangle = \hat{J}_1^2|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle\hat{J}_2^2|\uparrow\rangle + \hat{J}_{+1}|\uparrow\rangle\hat{J}_{-2}|\uparrow\rangle$$

$$+ \hat{J}_{-1}|\uparrow\rangle\hat{J}_{+2}|\uparrow\rangle + 2\hat{J}_z|\uparrow\rangle\hat{J}_z|\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle +$$

$$+0+0+2\cdot\frac{\hbar}{2}\cdot\frac{\hbar}{2}|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$ eigenstate of $\hat{J}^2$

$$\hat{J}^2|j,m\rangle = j(j+1)|j,m\rangle \Rightarrow |\uparrow\uparrow\rangle \equiv |j=1, m=1\rangle$$

State $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle \otimes |\downarrow\rangle$

$$\hat{J}^2|\uparrow\downarrow\rangle = \hat{J}^2|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle\hat{J}^2|\downarrow\rangle + \hat{J}_+|\uparrow\rangle\hat{J}_-|\downarrow\rangle + \hat{J}_-|\uparrow\rangle\hat{J}_+|\downarrow\rangle + 2\hat{J}_z|\uparrow\rangle J_z|\downarrow\rangle$$

$$= \frac{3\hbar^2}{4}|\uparrow\downarrow\rangle + \frac{3\hbar^2}{4}|\uparrow\downarrow\rangle + 0 + \underline{\hbar^2|\downarrow\uparrow\rangle} - \frac{\hbar^2}{2}|\uparrow\downarrow\rangle$$

$$\hat{J}^2|\downarrow\uparrow\rangle = \frac{3\hbar^2}{4}|\downarrow\uparrow\rangle + \frac{3\hbar^2}{4}|\downarrow\uparrow\rangle + \underline{\hbar^2|\uparrow\downarrow\rangle} + 0 - \frac{\hbar^2}{2}|\downarrow\uparrow\rangle$$

Individual states $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$ are <u>not</u> eigenstates of $\hat{J}^2$, but their sum or difference are!

$$\hat{J}^2\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)\right] = \underbrace{\left[\frac{3\hbar^2}{4}+\frac{3\hbar^2}{4}+\hbar^2-\frac{\hbar^2}{2}\right]}_{2\hbar^2}\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)$$

thus

$$\hat{J}^2\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)\right] = 2\hbar^2[\ldots] \Rightarrow j=1$$

$$\hat{J}_z\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)\right] = 0 \Rightarrow m=0$$

so this is the $|j=1, m=0\rangle$ state!

What about $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ state?

$$\hat{J}^2 \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \left[\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - \hbar^2 - \frac{\hbar^2}{2}\right][\ldots] = 0$$

eigenstate $\Downarrow$ with $j=0$ and $m=0$

$$|0,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

Alternative verification
(without using raising/lowering operators)

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ : $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$

$\hat{S}_z |\uparrow\uparrow\rangle = \hat{S}_{1z}|\uparrow\uparrow\rangle + \hat{S}_{2z}|\uparrow\uparrow\rangle = \frac{\hbar}{2}|\uparrow\uparrow\rangle + \frac{\hbar}{2}|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$

$\hat{S}_z |\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle$

$\hat{S}_z |\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\downarrow\rangle + \hat{S}_{2z}|\uparrow\downarrow\rangle = \frac{\hbar}{2}|\uparrow\downarrow\rangle - \frac{\hbar}{2}|\uparrow\downarrow\rangle = 0$

$\hat{S}_z |\downarrow\uparrow\rangle = 0$

So any combination of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue 0

$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = (\hat{S}_{1x} + \hat{S}_{2x})^2 + (\hat{S}_{1y} + \hat{S}_{2y})^2 + (\hat{S}_{1z} + \hat{S}_{2z})^2 =$

$= (\hat{S}_{1x}^2 + \hat{S}_{1y}^2 + \hat{S}_{1z}^2) + (\hat{S}_{2x}^2 + \hat{S}_{2y}^2 + \hat{S}_{2z}^2) + 2(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z})$

$= \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$

$\hat{S}^2|\uparrow\uparrow\rangle = \hat{S}_1^2|\uparrow\uparrow\rangle + \hat{S}_2^2|\uparrow\uparrow\rangle + 2(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z})|\uparrow\uparrow\rangle$

$= \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + 2\left[\cancel{\frac{\hbar^2}{4}|\downarrow\downarrow\rangle} - \cancel{\frac{\hbar^2}{4}|\downarrow\downarrow\rangle} + \frac{\hbar^2}{4}|\uparrow\uparrow\rangle\right] = 2\hbar^2|\uparrow\uparrow\rangle$

same for $|\downarrow\downarrow\rangle$

So $\hat{S}^2|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$ and $\hat{S}_z|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$

thus $|\uparrow\uparrow\rangle \equiv |1,1\rangle$ $\quad s=1, m=1$

Similarly $|\downarrow\downarrow\rangle \equiv |1,-1\rangle$ $\quad s=1, m=-1$

Brief reminder

$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \qquad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle \qquad \hat{S}_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

$\hat{S}_y |\uparrow\rangle = -\frac{i\hbar}{2} |\downarrow\rangle \qquad \hat{S}_y |\downarrow\rangle = \frac{i\hbar}{2} |\uparrow\rangle$

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$\hat{S}_{1x} \hat{S}_{2x} |\uparrow\downarrow\rangle = \frac{\hbar^2}{4} |\downarrow\uparrow\rangle \qquad \hat{S}_{1x} \hat{S}_{2x} |\downarrow\uparrow\rangle = \frac{\hbar^2}{4} |\uparrow\downarrow\rangle$

$\hat{S}_{1y} \hat{S}_{2y} |\uparrow\downarrow\rangle = \left(-\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{-2}\right) |\downarrow\uparrow\rangle = \frac{\hbar^2}{4} |\downarrow\uparrow\rangle \qquad \hat{S}_{1y} \hat{S}_{2y} |\downarrow\uparrow\rangle = \frac{\hbar^2}{4} |\uparrow\downarrow\rangle$

$\hat{S}_{1y} \hat{S}_{2y} |\uparrow\uparrow\rangle = \left(-\frac{i\hbar}{2}\right)^2 |\downarrow\downarrow\rangle = -\frac{\hbar^2}{4} |\downarrow\downarrow\rangle \qquad \hat{S}_{1y} \hat{S}_{2y} |\downarrow\downarrow\rangle = -\frac{\hbar^2}{4} |\uparrow\uparrow\rangle$

$\hat{S}_{1x} \hat{S}_{2x} |\uparrow\uparrow\rangle = \frac{\hbar^2}{4} |\downarrow\downarrow\rangle \qquad \hat{S}_{1x} \hat{S}_{2x} |\downarrow\downarrow\rangle = \frac{\hbar^2}{4} |\uparrow\uparrow\rangle$

$$\hat{S}_{1x}\hat{S}_{2x}\overbrace{\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)}^{|+\rangle} = \frac{\hbar^2}{4}\overbrace{\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle\right)}^{|+\rangle} = \frac{\hbar^2}{4}|+\rangle$$

$$\hat{S}_{1y}\hat{S}_{2y}|+\rangle = \frac{\hbar^2}{4}|+\rangle$$

$$\hat{S}_{1z}\hat{S}_{2z}|+\rangle = -\frac{\hbar^2}{4}|+\rangle$$

$$\hat{S}^2|+\rangle = \frac{3\hbar^2}{2}|+\rangle + 2\left[\frac{\hbar^2}{4}|+\rangle + \cancel{\frac{\hbar^2}{4}|+\rangle} - \cancel{\frac{\hbar^2}{4}|+\rangle}\right] = 2\hbar^2|+\rangle$$

Thus $\hat{S}^2|+\rangle = 2\hbar^2|+\rangle$ and $\hat{S}_z|+\rangle = 0$

and $|+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right) \equiv |1,0\rangle$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)$$

$$\hat{S}_{1x}\hat{S}_{2x}|-\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle-|\uparrow\downarrow\rangle\right) = -\frac{\hbar^2}{4}|-\rangle$$

$$\hat{S}_{1y}\hat{S}_{2y}|-\rangle = -\frac{\hbar^2}{4}|-\rangle$$

$$\hat{S}_{1z}\hat{S}_{2z}|-\rangle = -\frac{\hbar^2}{4}|-\rangle$$

$$\hat{S}^2|-\rangle = \frac{3\hbar^2}{2}|-\rangle + 2\left[\left(-\frac{\hbar^2}{4}\right)|-\rangle + \left(-\frac{\hbar^2}{4}\right)|-\rangle + \left(-\frac{\hbar^2}{4}\right)|-\rangle\right] = 0$$

So $\hat{S}^2|-\rangle = 0$ and $\hat{S}_z|-\rangle = 0$

thus $|-\rangle = |0,0\rangle$