The method for calculating the time evolution of the quantum states and expectation values

1. Find eigenstates and eigenvalues of the hamitonian

$\hat{H}|\psi_E\rangle = E|\psi_E\rangle$ Descrete spectrum

$E_1 \quad |\psi_1\rangle$
$E_2 \quad |\psi_2\rangle$
$\dots$

form complete basis

2. Decompose the initial state in the basis of the hamitonian eigenstates

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + c_3|\psi_3\rangle + \dots$$

3. Use the energy eigenvalues to add appropriate phase factors for each $|\psi_i\rangle$ state

$$|\psi(t)\rangle = c_1 e^{-iE_1t/\hbar}|\psi_1\rangle + c_2 e^{-iE_2t/\hbar}|\psi_2\rangle + \dots$$

4. Expectation values of even time-independent operators may change in time, if they do not commute with Hamiltonian

$$\frac{d}{dt}\langle A(t)\rangle = -\frac{1}{i\hbar}\langle[\hat{H},\hat{A}]\rangle$$

Practically

$$\langle A(t)\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \left(c_1^* e^{iE_1t/\hbar}\langle\psi_1| + c_2^* e^{iE_2t/\hbar}\langle\psi_2| + \dots\right)$$

$$\hat{A}\left(c_1 e^{-iE_1t/\hbar}|\psi_1\rangle + c_2 e^{-iE_2t/\hbar}|\psi_2\rangle + \dots\right)$$

Example 1: Precession of a Spin-1/2 particle in the magnetic field

Potential energy of a magnetic dipole in the magnetic field

$$U = -\vec{\mu}\cdot\vec{B} = -\mu_z \cdot B_z \quad \text{if} \quad \vec{B} = B_z \vec{k}$$

Hamiltonian $\quad \hat{H} = \hat{U} = -\hat{\mu}_z B_z = -\dfrac{g\cdot(-e)}{2mc}\hat{S}_z B_z$

$$\hat{H} = \left[\frac{ge}{2mc}B_z\right]\hat{S}_z = \omega_0 \hat{S}_z \qquad \omega_0 = \frac{ge}{2mc}B_z$$

Larmor frequency

Eigenstates of $\hat{H} = \omega_0 \hat{S}_z \qquad |\pm z\rangle$

Eigen values $\quad \hat{H}|+z\rangle = \underbrace{\dfrac{\hbar\omega_0}{2}}_{E_+}|+z\rangle \qquad \hat{H}|-z\rangle = \underbrace{-\dfrac{\hbar\omega_0}{2}}_{E_-}|-z\rangle$

$t=0$

$$|\psi(t=0)\rangle = |+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle) \Rightarrow \quad |\psi(t)\rangle - ?$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|+z\rangle + \frac{1}{\sqrt{2}}e^{i\omega_0 t/2}|-z\rangle$$

$\omega_0 t = 0 \qquad |\psi(t=0)\rangle = \dfrac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle) = |+x\rangle$

$\omega_0 t = \pi \qquad |\psi(t=\tfrac{\pi}{\omega_0})\rangle = \dfrac{1}{\sqrt{2}}\left(\underbrace{e^{-\frac{i\pi}{2}}}_{-i}|+z\rangle + \underbrace{e^{+i\pi/2}}_{i}|+z\rangle\right) =$

$$|\psi(t=\pi/\omega_0)\rangle = -\frac{i}{\sqrt{2}}(|+z\rangle - |-z\rangle) = -i\,|-x\rangle$$

$\omega_0 t = \pi/2 \qquad |\psi(t=\pi/2\omega_0)\rangle = \dfrac{1}{\sqrt{2}}\left(e^{-i\pi/4}|+z\rangle + e^{i\pi/4}|-z\rangle\right) =$

$$= e^{-i\pi/4}\frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} |+z\rangle + \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} |-z\rangle$$

$$\langle S_z\rangle(t) = \langle\psi(t)|\hat{S}_z|\psi(t)\rangle = \left\langle \frac{1}{\sqrt{2}} e^{+i\omega_0 t/2} \langle +z| + \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} \langle -z| \hat{S}_z |\psi(t)\rangle\right.$$

$$= \frac{1}{2}\left[\langle +z|\hat{S}_z|+z\rangle + \langle -z|\hat{S}_z|-z\rangle\right] = \frac{1}{2}\left[\frac{\hbar}{2} - \frac{\hbar}{2}\right] = 0$$

does not change in time

What about other components?

$$\langle S_x\rangle(t) = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\omega_0 t/2}\langle +z| + e^{-i\omega_0 t/2}\langle -z|\right)\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+z\rangle + e^{+i\omega_0 t/2}|-z\rangle\right)$$

$$= \frac{1}{2}\frac{\hbar}{2}\left(e^{i\omega_0 t/2} \quad e^{-i\omega_0 t/2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2} \end{pmatrix} =$$

$$= \frac{\hbar}{4}\left(e^{i\omega_0 t/2} \quad e^{-i\omega_0 t/2}\right)\begin{pmatrix} e^{i\omega_0 t/2} \\ e^{-i\omega_0 t/2} \end{pmatrix} =$$

$$= \frac{1}{2}\frac{\hbar}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) = \frac{\hbar}{2}\cos\omega_0 t$$

Following same steps

$$\langle S_y\rangle(t) = \frac{\hbar}{2}\sin\omega_0 t$$

So the average transverse spin rotates in x-y plane with ω_0 – Larmor precession, in full accord with classical expectations

Note that this result matches the prediction $\frac{d}{dt}\langle\hat{A}(t)\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$

indeed $[\hat{H},\hat{S}_x] = \omega_0 [\hat{S}_z,\hat{S}_x] = i\hbar\omega_0 \hat{S}_y$

$[\hat{H},\hat{S}_y] = \omega_0 [\hat{S}_z,\hat{S}_y] = -i\hbar\omega_0 \hat{S}_x$

$$\frac{d}{dt}\langle\hat{S}_x\rangle = \frac{i}{\hbar} i\hbar\omega_0\langle\hat{S}_y\rangle = -\omega_0\langle\hat{S}_y\rangle$$

$$\frac{d}{dt}\langle\hat{S}_y\rangle = \frac{i}{\hbar}(-i\hbar\omega_0)\langle\hat{S}_x\rangle = \omega_0\langle\hat{S}_x\rangle$$

$$\frac{d^2}{dt^2}\langle\hat{S}_x\rangle = -\omega_0\langle\hat{S}_x\rangle$$ spin precession