Steps to solve time-evolution problems

1. Find eigenstates and eigenvalues of the Hamiltonian (stationary states)

2. Decompose the initial quantum state in the basis of the stationary states

3. The final (time-evolved) state is the same superposition with each stationary state evolving as $e^{-iEt/\hbar}$

4. Make the desired measurement (or calculate the mean operator value) using the final state

The example solution using 4.12

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = \omega_0 \hat{S}_x$$

1. Eigenstates $|S_x = +\hbar\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ +\sqrt{2} \\ 1 \end{pmatrix}$ $\quad |S_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$|S_x = -\hbar\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$$\hat{H}|S_x = +\hbar\rangle = \hbar\omega_0 |S_x = +\hbar\rangle \qquad \hat{H}|S_x = 0\rangle = 0$$

$$\hat{H}|S_x = -\hbar\rangle = -\hbar\omega_0 |S_x = -\hbar\rangle$$

2. Initial state $|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1,1\rangle\big(|S_x = \hbar\rangle\langle S_x = \hbar| +$

$+|S_x = 0\rangle\langle S_x = 0| + |S_x = -\hbar\rangle\langle S_x = -\hbar|\big) = |1,1\rangle\langle S_x = \hbar|\,|S_x = \hbar\rangle + \ldots$

$+|1,1\rangle\langle S_x = 0|\,|S_x = 0\rangle + |1,1\rangle\langle S_x = -\hbar|\,|S_x = -\hbar\rangle = \frac{1}{2}|S_x = \hbar\rangle + \frac{1}{\sqrt{2}}|S_x = 0\rangle$

$+\frac{1}{2}|S_x = -\hbar\rangle$

3. Time-evolution : $|fin\rangle = \frac{1}{2} e^{-i\omega_0 t} |S_x = \hbar\rangle +$

$+ \frac{1}{\sqrt{2}} |S_x = 0\rangle + \frac{1}{2} e^{i\omega_0 t} |S_x = -\hbar\rangle$

4. Final measurement : probability to be @ $|1,-1\rangle$ state

$$P = |\langle 1,-1|fin\rangle|^2$$

$$\langle 1,-1|fin\rangle = (0 \;\; 0 \;\; 1)\left[\frac{1}{2} e^{-i\omega_0 t} \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2} e^{i\omega_0 t} \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right.$$

$$= \frac{1}{4} e^{-i\omega_0 t} - \frac{1}{2} + \frac{1}{4} e^{i\omega_0 t} = \frac{1}{2}\cos\omega_0 t - \frac{1}{2} = -\sin^2 \frac{\omega_0 t}{2}$$

$$P = |\langle 1,-1| fin\rangle|^2 = \sin^4 \frac{\omega_0 t}{2}$$