# Photon wave function

Classical description of a propagating electromagnetic wave

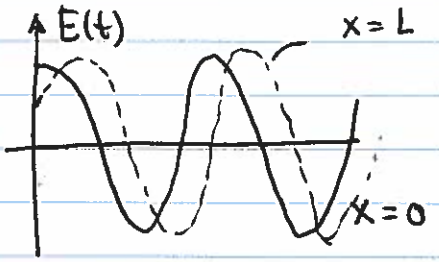
$E = E_0 \cos(kx - \omega t)$

as the wave propagates, it keeps the same form, but aquire a phase

at $x = 0$ $\quad E(t) = E_0 \cos(\omega t)$

at $x = L$ $\quad E(t) = E_0 \cos(kL - \omega t) = E_0 \cos(\omega t - \varphi)$

$\varphi = k \cdot L = 2\pi L/\lambda$




Same oscillation, but shifted by $\varphi = 2\pi L/\lambda$

A wave function describing a plane wave photon $\psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{ikx - i\omega t}$

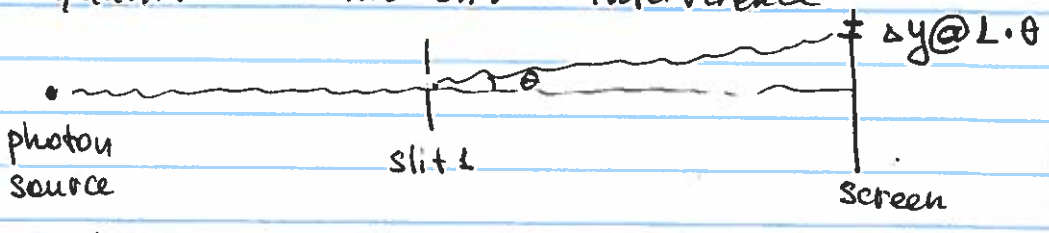
here the photon moves along x-axis, thus $e^{ikx}$

for a photon moving to an opposite direction $\psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{-ikx - i\omega t}$

When the photon travels, it also develops the phase

$\psi(x,t) \rightarrow \psi(x+L,t) = e^{ikL} \psi(x,t)$

Quantum two-slit interference




Let is say the probability of a photon to pass through slit 1 is $p_1$

Photon wave function at a slit $\psi_1 = \sqrt{p_1}$

Then it travel distance $L/\cos\theta \approx L$ (for small $\theta$) to the screen

$\psi_1$ at the screen $= \sqrt{p_1}\, e^{ikL/\cos\theta} \approx \sqrt{p}\, e^{ikL}$

Probability of detecting a photon in a small section $\Delta y$ at the distance $y = L\theta$: $|\psi_1(\theta)|^2 \cdot \Delta y = (p_1)\underbrace{|e^{ikL/\cos\theta}|^2}_{=1} \cdot \Delta y$

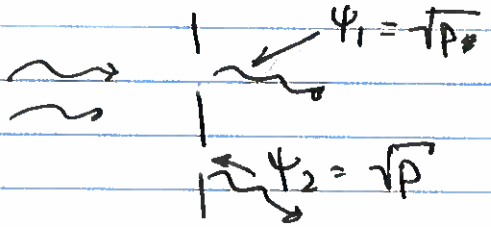
$= p_1 \cdot \Delta y$

Basically we can detect the photon anywhere within the laser spot with equal probability.

If there is another slit, shifted by d vertically, then the situation is exactly the same — one big blob

Even if we have both slits there, if we know through which slit photon travels, we are not going to see any interference

Now imagine we don't know through which slit the photon went




Total wavefunction

$\psi = \psi_1 + \psi_2$
slits

Just as in classical treatment, we know that the optical path ~~from~~ two slits to the screen is almost the same, except for an extra distance $\Delta x = d \cdot \theta$ (or $d \cdot \sin\theta$ in general)

So at the screen

$\psi_{screen} = \sqrt{p}\, e^{ikL} + \sqrt{p}\, e^{ik(L+\Delta x)} =$

$= \sqrt{p}\, e^{ikL} \underbrace{(1 + e^{ik\Delta x})}$ interference term

Probability ~~of detection~~

if $k\Delta x = \frac{2\pi}{\lambda} \cdot d \cdot \theta = 0, 2\pi, 4\pi \ldots \pm 2\pi m$

$e^{i \cdot 2\pi m} = 1$ $\quad \psi_{screen} = 2\sqrt{p}\, e^{ikL}$

detection probability ~~to be~~ $\propto |\psi|^2$

$|\psi_{screen}|^2 \cdot \Delta y = 4p \cdot \Delta y$ (constructive interference)

if $k\Delta x = \frac{2\pi}{\lambda} d\theta = \pm\pi, \pm 3\pi \ldots$ $\quad e^{\pm i\pi} = -1$

$\psi_{screen} = 0$

Probability of detection is zero

In general probability of detection

$$\left| \tfrac{1}{\sqrt{2}} e^{ikL} \left(1 + e^{ik\Delta x}\right) \right|^2 \Delta y = p\Delta y \left| 1 + e^{ik\Delta x} \right|^2 =$$

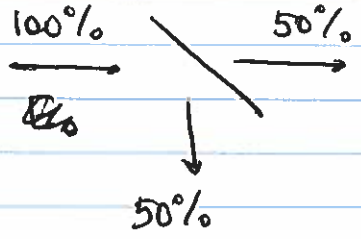
$$= p\Delta y \left| e^{ik\Delta x/2} \left( e^{-ik\Delta x/2} + e^{ik\Delta x/2} \right) \right|^2 =$$

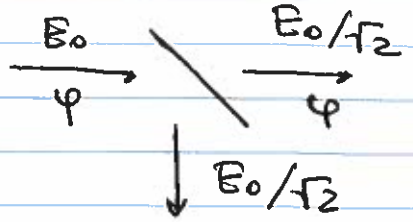
$$= p\Delta y \left| e^{ik\Delta x/2} \right|^2 \left| 2\cos\frac{k\Delta x}{2} \right|^2 = 4p\Delta y \cos^2\frac{k\Delta x}{2} =$$

$$= 4p\Delta y \cos^2\left(\frac{\pi \cdot d \cdot \theta}{\lambda}\right)$$

To summarize: if we know through which slit the photon traveled → no interference
If we don't → interference
Bizarrely, we don't need to know, as soon as anyone (or anything) conducts the measurement, and photon wavefunction is collapsed, and interference disappears.

# Light at a beamsplitter




in laser power / energy

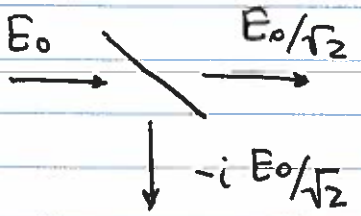
phase shift

Incoming wave: $E_0 \cos(kx - \omega t)$

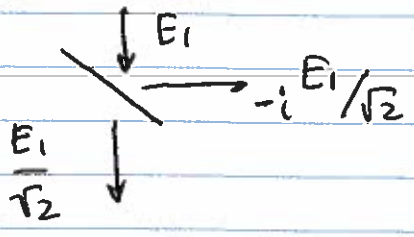
Transmitted wave: $\underbrace{1/\sqrt{2}}_{\sqrt{50\%}\text{ ← transmitton probability.}} E_0 \cos(kx - \omega t)$

Reflected wave: $\underbrace{1/\sqrt{2}\, E_0}_{\sqrt{50\%}\text{ ← reflection probability}} \sin(kx - \omega t) = \frac{1}{\sqrt{2}} E_0 \cos(kx - \omega t - \frac{\pi}{2})$

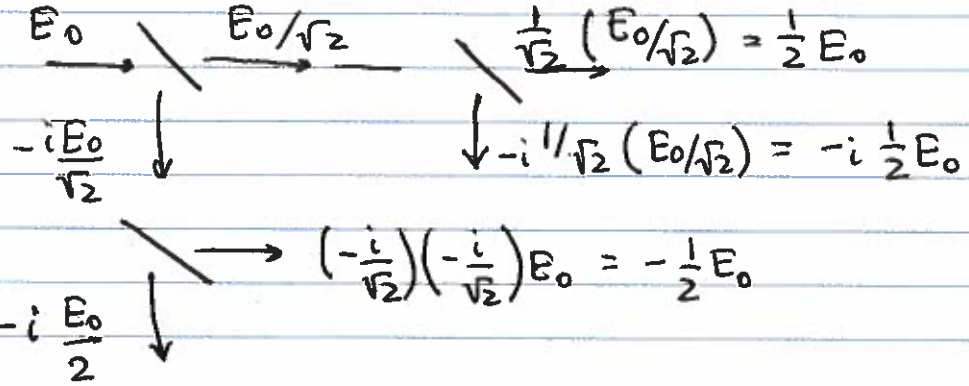
We can use a complex phase to describe



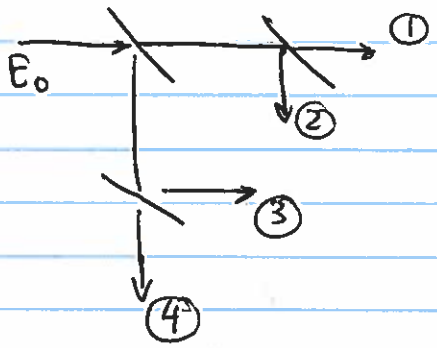

and




# Cascading beam splitters



Four possible outcomes




$E_1 = \frac{1}{2} E_0$ (in phase)

$E_2 = -\frac{i}{2} E_0$ } 90° phase shifted

$E_4 = -\frac{i}{2} E_0$ }

$E_3 = -\frac{1}{2} E_0$ (out of phase)

If a single photon will enter, its wave function will contain 4 components, corresponding to four possible locations

$$\Psi_{out} = \frac{1}{2}\Psi_1 + \frac{i}{2}\Psi_2 - \frac{i}{2}\Psi_3 - \frac{1}{2}\Psi_4$$

Of course, if we put single-photon detectors at each output port, we only measure it in one of them, with equal probabilities.