Quantization of electromagnetic field
(abbreviated story)

What we know: light behaves as a wave
(it can interfere and diffract)
light also behave as a particle
(photo electric effect, energy quantization)

Classical description
$\vec{E} = \vec{E}_0 \cos(kz - \omega t)$ the fact that electric fields add up as vectors ~~a~~ is responsible for interference.

Energy of the e-m field

$$\mathcal{E} = \frac{1}{2}\int dV \left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H} \right) = \cancel{\tfrac{1}{2}}\varepsilon_0 \int dV \cdot E^2 = \frac{\varepsilon_0}{2} V \cdot E_0^2$$

When we measure the energy, it must be quantized.

If we have a quantum state with known number of photons $|n\rangle$

$$\hat{\mathcal{E}}\,|n\rangle = \hbar\omega \cdot n\,|n\rangle$$

Since classical and quantum expressions must match

$$\hbar\omega n \Longleftrightarrow \frac{\varepsilon_0}{2} V \cdot E_0^2 \qquad E_0 \propto \sqrt{n}$$

$n=1$ $\hbar\omega = \frac{V}{2}\varepsilon_0 E_0^2$ $E_0 = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 \cdot V}}$ electric field of a single photon.

Number operator $\hat{n}|n\rangle = n|n\rangle$

but how to take a square?!

Quantum operators of creation and annihilation

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ one photon is destroyed
(annihilation)

$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ one photon is added
(creation)

These operators originates from describing e-m field as an oscillator

$\hat{a}^+\hat{a}|n\rangle = \hat{a}^+\sqrt{n}\,|n-1\rangle = n\,|n\rangle$

So $\hat{n} = \hat{a}^+\hat{a}$ (not quite $\hat{n} \sim \hat{a}^2$, but close)

Quantum operator for e-m field

$$\hat{E} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\vec{e}_p\left(\hat{a}\,e^{-i\omega t+ikz} + \hat{a}^+ e^{i\omega t-ikz}\right)u(x,y)$$

$\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$ - electric field of a single photon

$\vec{e}_p$ - polarization vector

$u(x,y)$ - transverse intensity distribution (if different from a plane wave)

$\hat{a}$ & $\hat{a}^+$ - annihilation and creation operators

Electro-magnetic wave is still a wave, but it allows only discrete number of excitations

Let's calculate a proper expression for e-m wave energy

$$\hat{\mathcal{E}} = \varepsilon_0 \int dV \cdot E^2 = \left\{ \begin{matrix} \text{assuming} \\ \text{plane} \\ \text{wave} \end{matrix} \right\} = \varepsilon_0 V \cdot \hat{E}^2$$

$$\hat{\mathcal{E}} = \frac{\hbar\omega}{2\cancel{\varepsilon_0 V}} \cdot \cancel{\varepsilon_0 V} \cdot \Big(\underbrace{\cancel{\hat{a}\hat{a}\, e^{-2i\omega t + 2ikz}} + \cancel{\hat{a}^+\hat{a}^+ e^{+2i\omega t - 2ikz}}}_{\text{fast-oscillating terms}} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} \Big)$$

(automatically disappear with more accurate derivations)

$$\hat{\mathcal{E}} = \frac{\hbar\omega}{2} \left(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+ \right)$$

Very important property of QM operators: Order matter a lot!

$$\underbrace{\hat{a}^+\hat{a}}_{\hat{n}} |n\rangle = n|n\rangle \qquad \hat{a}\hat{a}^+ |n\rangle = \hat{a}\left(\sqrt{n+1}\,|n+1\rangle\right) = (n+1)|n\rangle = n|n\rangle + |n\rangle = (\hat{a}^+\hat{a} + 1)|n\rangle$$

$$\hat{\mathcal{E}}|n\rangle = \frac{\hbar\omega}{2}\left(2\hat{a}^+\hat{a} + 1\right)|n\rangle = \hbar\omega\left(\hat{n} + \tfrac{1}{2}\right)|n\rangle$$

For a state with known number of photons

$$\mathcal{E}_n \doteq \hat{\mathcal{E}}|n\rangle = \hbar\omega\left(\hat{n} + \tfrac{1}{2}\right)|n\rangle = \underbrace{\hbar\omega\left(n + \tfrac{1}{2}\right)}_{\mathcal{E}_n}|n\rangle$$

$$\mathcal{E}_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

For $n=0$ (vacuum field) $\mathcal{E}_n = \frac{1}{2}\hbar\omega \neq 0$!
Vacuum has energy!

These vacuum fluctuations case spontaneous emission of electrons in excited states