Classical model of Resonance Absorption and Dispersion

Vacuum: $\vec{E}(z,t) = \vec{E}_0 \cos(k_0 z - \omega t)$ $\quad k_0 = \frac{2\pi}{\lambda_0}$

Electromagnetic wave in a material

$$\vec{E} = \vec{E}_0 \cos(kz - \omega t) = \vec{E}_0 \cos\left(\frac{2\pi n}{\lambda_0} z - \omega t\right)$$

Up to now we considered n to be a real number. But it is not always the case! Let's use complex form of e-m wave to see more easily ~~what~~ what a complex refractive index value means

$$\vec{E} = Re\left\{\vec{E}_0 e^{i\left(\frac{2\pi}{\lambda_0} \cdot nz - \omega t\right)}\right\} = Re\left\{\vec{E}_0 e^{i\left(\frac{2\pi}{\lambda_0}(n' + in'')z - \omega t\right)}\right\}$$

$$= Re\left\{\vec{E}_0 e^{i\left(\frac{2\pi n'}{\lambda_0} z - \omega t\right)}\right\} \cdot e^{-\frac{2\pi}{\lambda_0} n'' z} = \vec{E}_0 \cos\left(\frac{2\pi n'}{\lambda_0} z - \omega t\right) e^{-\frac{2\pi}{\lambda_0} n'' z}$$

Intensity $\quad I = \frac{1}{c} \langle E^2 \rangle = \frac{1}{2c} E_0^2 \; e^{-\frac{4\pi n''}{\lambda_0} \cdot z}$

$$\beta = \frac{4\pi n''}{\lambda_0} \quad \text{— absorption coefficient}$$

So the <u>real part</u> of the refractive index n' is responsible for speed of e-m wave being different in a material $v = c/n'$

Imaginary part n'' describes the absorption

We first model a dialectric as an electron on a spring

$\bar{e}$ $\downarrow F = -kx$ elastic force
(Coulumb force in the case of an electron)

nuclei

Resonance frequency for an atom (no external fields)

$ma = F$ $\qquad m \cdot \frac{d^2x}{dt^2} = -kx \rightarrow$ oscillations

$x = x_0 \cos \omega_0 t$

$$\boxed{\omega_0 = \sqrt{k/m}}$$

very crude model, but gives a general idea

With external electromagnetic field

$\vec{E} = \vec{E}_0 \cos(\underbrace{kz}_{} - \omega t)$

atom is small, $\vec{E}$ does not change across an atom

$\Downarrow$

atom "sees" a time-dependent electric field $\vec{E} = E_0 \cos(\omega t)$, $\vec{F} = -e \cdot \vec{E}$

$$m \frac{d^2x}{dt^2} = \underbrace{-eE_0 \cos \omega t}_{\text{external field}} - \underbrace{kx}_{\text{Coulumb force}} - \underbrace{\xi (dx/dt)}_{\text{friction, viscosity}}$$

describes finite lifetime of ~~ato~~ any electron oscillation

It is easier to solve in a complex form again

$$m \frac{d^2x}{dt^2} = -eE_0 e^{i\omega t} - kx - \xi \frac{dx}{dt}$$

forced oscillations

$x = a e^{i\omega t}$

$$m a(-\omega^2) e^{i\omega t} = -eE_0 e^{i\omega t} - k\cdot a e^{i\omega t} - \xi(i\omega) a e^{i\omega t}$$

$$eE_0 = a(m\omega^2 - m\omega_0^2 - i\xi\omega) = am(\omega^2-\omega_0^2 - i\omega\gamma)$$

$$\gamma = \xi/m$$

$$a = \frac{-eE_0}{m}\ \frac{1}{\omega_0^2-\omega^2+i\gamma\omega}$$

$$x = a e^{i\omega t} = \left(\frac{-eE_0}{m}\right)\frac{1}{\omega_0^2-\omega^2+i\gamma\omega}\ e^{i\omega t}$$

Complex amplitude means the electron oscillates not quite in phase with the electric field ("friction" slowes it down)

Remember polarization of an atom?

$$\vec{P} = \alpha \vec{E}_0 e^{i\omega t} = \underbrace{(-e)\cdot x}_{\text{dipole moment by definition}} = \underbrace{\boxed{\frac{e^2}{m}\ \frac{1}{\omega_0^2-\omega^2+i\gamma\omega}}}_{\alpha}\ E_0 e^{i\omega t}$$

Just as we did for the calculations of scattering

$$\varepsilon = 1 + N\cdot\alpha/\varepsilon_0$$

$$n = \sqrt{\varepsilon} \approx 1 + \tfrac{1}{2} N\alpha/\varepsilon_0$$

$$n = 1 + \frac{1}{2}\underbrace{\boxed{\frac{e^2}{\varepsilon_0 m}\cdot N}}_{\omega_p^2}\ \frac{1}{\omega_0^2-\omega^2+i\gamma\omega}$$

Typically $\gamma \ll \omega_0$

$\omega_p^2 = N\dfrac{e^2}{\varepsilon_0 m}$ - value called plasma frequency is a material property parameter

If ω and ω_0 are different — n does not change much with ω - off-resonant interaction

The strongest interaction occures when

$\omega \approx \omega_0$ $\qquad \omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) \approx 2\omega(\omega_0-\omega)$

$$\boxed{n \simeq 1 + \frac{\omega_p^2}{4\omega} \frac{1}{\omega_0 - \omega + i\gamma/2}}$$

This is a correct functional form of resonant absorption!

Real part of the refractive index

$$n' = 1 + \frac{\omega_p^2}{4\omega} \, Re\left[\frac{(\omega_0-\omega) - i\gamma/2}{(\omega_0-\omega)^2 + \gamma^2/4} \right] =$$

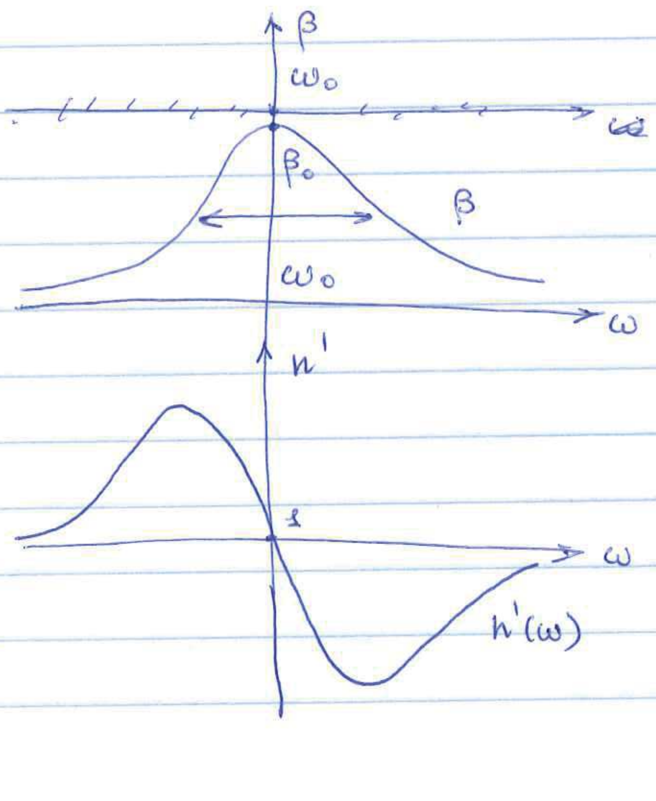
$$= 1 + \frac{\omega_p^2}{4\omega} \frac{\omega_0 - \omega}{(\omega_0-\omega)^2 + \gamma^2/4}$$

Imaginary part of n

$$n'' = \frac{\omega_p^2}{8\omega} \frac{\gamma}{(\omega_0-\omega)^2 + \gamma^2/4}$$

Absorption coefficient $\beta = \frac{2\pi}{\lambda_0} n''$ is proportional to the "friction" coefficient — γ is the measure of dissipation of energy in our atom

Strongest absorption — $\omega = \omega_0$



$$\beta_0 = \frac{2\pi}{\lambda_0} \frac{\omega_p^2}{2\omega_0 \gamma}.$$

resonance absorption

Resonance linewidth $\sim \gamma$

$n'(\omega = \omega_0) = 1$

What is the velocity of light

c – speed of light in vacuum

$v = c/n$ phase velocity in a material

phase front speed in a constant e-m wave

What if ~~one~~ ~~the~~ light is not constant, but propagates as a short pulse?

In a short pulse one has to take into account ~~co~~ different frequency components

$$E(z,t) = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} E(\omega)\, e^{i k(\omega)\cdot z - i\omega t}\, d\omega$$

$$k(\omega) = \frac{2\pi}{\lambda_0} n'(\omega) \leftarrow \text{not a constant!}$$

Typically, the spread of frequencies in a pulse is much smaller than the width of an atomic resonance, so we can use a linear approximation for $k(\omega)$

$$k(\omega) = k(\omega_0) + \frac{d\omega}{dk}\cdot\Delta\omega$$

$$E(z,t) \simeq \cancel{E_{ao}}\; e^{i k(\omega_0) z - i\omega_0 t} \underbrace{\int_{-\Delta\omega}^{\Delta\omega} E_0(\Delta\omega)\, e^{i \frac{d\omega}{dk}\Delta\omega z - i\Delta\omega t}\, d\omega}_{\text{describes the field envelope propagation}}$$

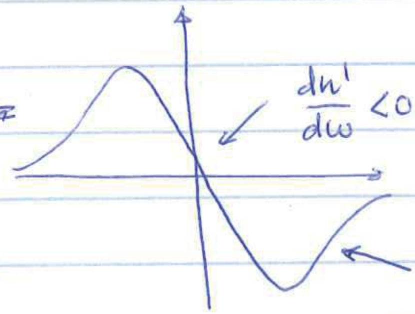
Group velocity $v_g = \dfrac{d\omega}{dk}$

One can show that the energy of em wave propagates with the group velocity

For our resonance

$$n' = 1 + \frac{\omega_p^2}{4\omega} \frac{\omega_0 - \omega}{(\omega_0-\omega)^2 + \gamma^2/4} \qquad k = \frac{\omega}{c} \cdot n$$

$$v_g = 1/\frac{dk}{d\omega} = \frac{1}{n/c + \frac{\omega}{c}\frac{dn'}{d\omega}} = \frac{c}{n + \omega \frac{dn'}{d\omega}}$$



$\frac{dn'}{d\omega} < 0$ on resonance
negative dispersion

$\frac{dn'}{d\omega} > 0$ on the wings
of the resonance
positive dispersion

On the absorption resonance $\omega = \omega_0$

$$v_g = \frac{c}{1 - \left|\omega \frac{dn'}{d\omega}\right|} > c$$

superluminal light propagation

On the wings

$$v_g = \frac{c}{1 + \left|\omega \frac{dn'}{d\omega}\right|} < c$$

subluminal light propagation
slow light