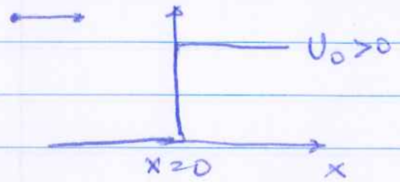
Potential steps and barriers

Above-barrier behavior $E > U_0$

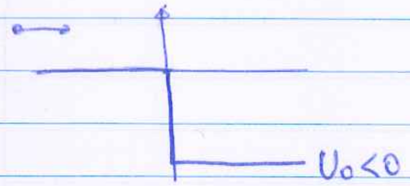
Potential step



Classical particle travels through, but move slower for $x > 0$

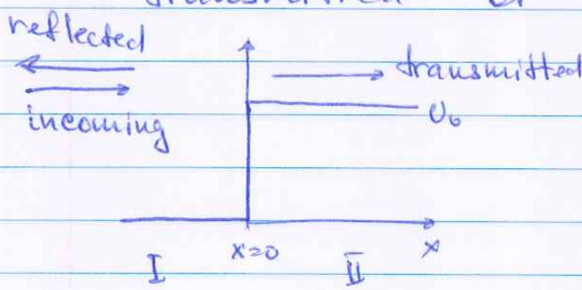
$K_1 = E$ for $x < 0$

$K_2 = E - U_0$ for $x > 0$



if it is a step down, the particle would move faster for $x > 0$ $\quad K_2' = E + |U_0|$

Quantum particle can be ~~to~~ either being transmitted or reflected



$x < 0: \psi_I = \underbrace{Ae^{ik_1 x}}_{\text{incoming}} + \underbrace{Be^{-ik_1 x}}_{\text{reflected}}$ $\qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$x > 0: \psi_{II} = \underbrace{Ce^{ik_2 x}}_{\text{transmitted}}$ $\qquad k_2 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

Boundary conditions

$\psi_I(0) = \psi_{II}(0) \qquad A + B = C$

$\psi_I'(0) = \psi_{II}'(0) \qquad ik_1(A-B) = ik_2 C = ik_2(A+B)$

$$(k_1 - k_2)A = (k_1 + k_2)B$$

Reflection ~~coe~~ probability (same as reflection coefficient)

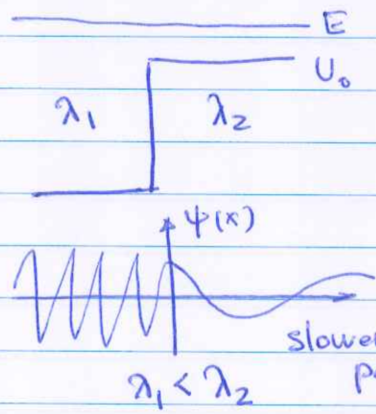
$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Transmission probability $\quad T = 1 - R$

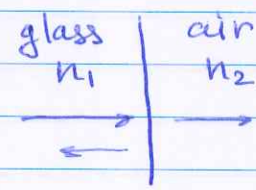
Such reflection off a low barrier seem counter intuitive for particle, but completely normal for waves, as we always have reflection on a boundary b/w transparent materials

Recall that for a free moving particle in a state described by $\psi = e^{\pm ikx}$
$p_x = \pm \hbar k$, or $p = \hbar k$, and thus
$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi}{k} \quad \text{deBroigle wavelength}$$

at the same time the kinetik energy
$$K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$
so a particle with higher kinetic energy corresponds to a shorter wavelength



$\Longleftrightarrow$

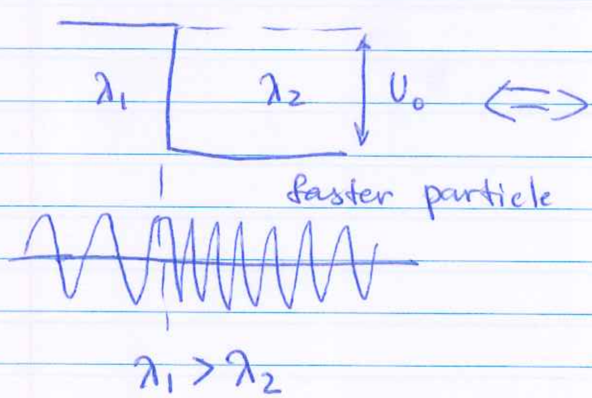


$n_2 > n_1$

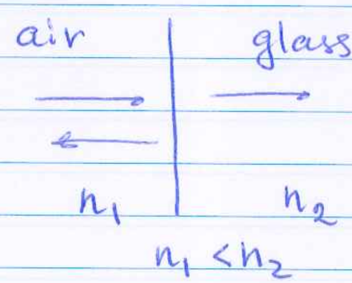
in optics $\lambda_n = \frac{\lambda_0}{n}$ ← in vacuum

$$k = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\lambda_0} \cdot n$$

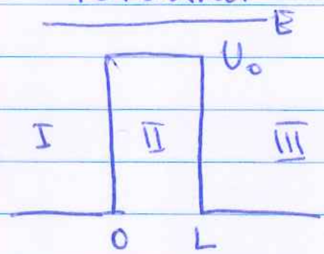
for



$\Longleftrightarrow$



Potential barrier



$E > U_0$

$$\psi_I(x) = \underbrace{Ae^{ikx}}_{\text{incoming}} + \underbrace{Be^{-ikx}}_{\text{reflected}} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi_{III}(x) = \underbrace{Ce^{ikx}}_{\text{transmitted}}$$

$$\psi_{II}(x) = De^{ik_1x} + Fe^{-ik_1x} \qquad k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

Boundary conditions

$x = 0$

$\psi_I(0) = \psi_{II}(0)$ $\qquad A + B = D + F \quad (1)$

$\psi_I'(0) = \psi_{II}'(0)$ $\qquad ik(A-B) = ik_1(D-F) \quad (2)$

$x = L$

$\psi_{II}(L) = \psi_{III}(L)$ $\qquad De^{ik_1L} + Fe^{-ik_1L} = Ce^{ikL} \quad (3)$

$\psi_{II}'(L) = \psi_{III}'(L)$ $\qquad ik_1(De^{ik_1L} - Fe^{-ik_1L}) = ikCe^{ikL} \quad (4)$

Now we need to solve for $\frac{C}{A}$ to find transmission probability or B/A to find the reflection probability

(1) $\quad B = D + F - A$

(3+4) $\quad ik_1(De^{ik_1L} - Fe^{-ik_1L}) = ik(De^{ik_1L} + Fe^{-ik_1L})$

$$F = \frac{k_1 - k}{k_1 + k} e^{2ik_1L} D$$

(2) $$2kA - (k+k_1)D = -\frac{(k_1-k)^2}{k+k_1} e^{2ik_1L} D$$

$$D = \frac{2k(k+k_1)A}{(k+k_1)^2 - (k-k_1)^2 e^{2ik_1L}}$$

$$F = -A\frac{(k-k_1)}{(k+k_1)} e^{2ik_1L} D = -\frac{2k(k-k_1)A}{(k+k_1)e^{-2ik_1L} - (k-k_1)^2}$$

$$Ce^{ikL} = \frac{4kk_1 A}{(k+k_1)^2-(k-k_1)^2 e^{2ik_1L}} = \frac{2k_1k\, e^{ik_1L} A}{2kk_1\cos k_1L - i(k^2+k_1^2)\sin k_1L}$$

$$C = A\,\frac{2k_1k \quad e^{ik_1a-ika}}{2k_1k\cos k_1L - i(k^2+k_1^2)\sin k_1L}$$

Transmission probability $T = \left|\frac{C}{A}\right|^2$

$$T = \frac{4k_1^2k^2}{4k_1^2k^2\cos^2k_1L + (k^2+k_1^2)\sin^2k_1L} = \frac{1}{1+\frac{(k^2-k_1^2)^2}{4k_1^2k^2}\sin^2k_1L}$$

since $k^2 = \frac{2mE}{\hbar^2}$ $\quad k_1^2 = \frac{2m(E-U_0)}{\hbar^2}$

$$T(E) = \frac{1}{1+\frac{U_0^2}{4E(E-U_0)}\sin^2k_1L} \qquad E>U_0$$

reminder

$$T(E) = \frac{1}{1+\frac{U_0^2}{4E(U_0-E)}\sinh^2kL} \qquad E<U_0$$