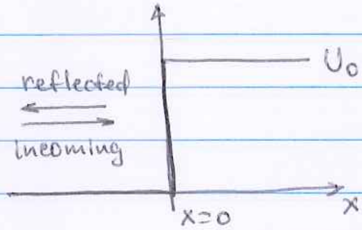
Reflection of a potential step



Classically allowed region $x<0$
incoming particle with energy E
(any energy is possible now)
reflected particles will have the same energy E, but will move in an opposite direction

$$\psi_{x<0}(x) = \underbrace{A e^{ikx}}_{\text{incoming}} + \underbrace{B e^{-ikx}}_{\text{reflected}}$$

$x>0$ Classically forbidden region $E<U_0$ $\rightarrow$ exponential solution

$$\psi_{x>0}(x) = \underbrace{\cancel{Ce^{\alpha x}}}_{\text{blows up}} + De^{-\alpha x}$$

To find the ratios between the coefficients, we use the boundary conditions: $\psi(x)$ is continuous and smooth at $x=0$

cont: $A+B=D$

smooth $\psi'_{x<0}(0) = \psi'_{x>0}(0)$

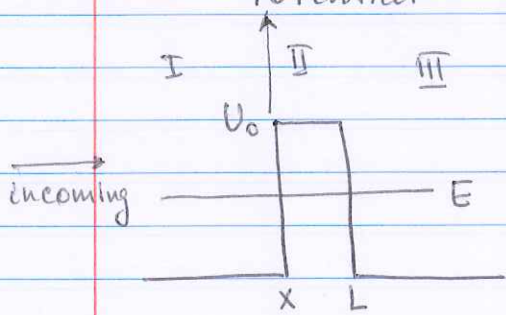
$$ikA - ikB = -\alpha D$$
$$ik(A-B) = -\alpha(A+B)$$
$$(ik+\alpha)A = (ik-\alpha)B$$
$$\frac{B}{A} = \frac{ik-\alpha}{ik+\alpha}$$

Reflection probability $R = \left|\frac{B}{A}\right|^2 = \left|\frac{ik-\alpha}{ik+\alpha}\right|^2 = 1$

100% reflection probability

(analogous to a 100% mirror, reflecting laser beam)

Potential barrier



$$U(x) = \begin{cases} 0 & x<0,\ x>L \\ U_0 & 0<x<L \end{cases}$$

if $E<U_0$

classically allowed regions (I, III)

$x<0 \quad x>L$

classically forbidden region (II)

$0<x<L$

Region I ($x<0$): incoming and reflected beams

$$\psi_I(x) = \underbrace{A e^{ikx}}_{\text{incoming}} + \underbrace{B e^{-ikx}}_{\text{reflected}} \qquad k=\sqrt{\frac{2mE}{\hbar^2}}$$

Region III ($x>L$): transmitted beam

$$\psi_{III}(x) = C e^{ikx}$$

Region II ($0<x<L$): classically forbidden region

$$\psi_{II}(x) = F e^{-\alpha x} + G e^{\alpha x} \qquad \alpha = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

Reflection probability $R = \left|\frac{B}{A}\right|^2$

Transmission probability $T = \left|\frac{C}{A}\right|^2$

To find this ratios, we need to use the boundary conditions

$x=0$ $\quad \psi_I(0) = \psi_{II}(0) \quad$ (1) $A+B = F+G$

$\quad \psi_I'(0) = \psi_{II}'(0) \quad$ (2) $ikA - ikB = -\alpha F + \alpha G$

$x=L$ $\quad \psi_{II}(L) = \psi_{III}(L) \quad$ (3) $Fe^{-\alpha L} + Ge^{\alpha L} = Ce^{ikL}$

$\quad \psi_{II}'(L) = \psi_{III}'(L) \quad$ (4) $-\alpha F e^{-\alpha L} + \alpha G e^{\alpha L} = ikCe^{ikL}$

To find the transmission probability we need to find the ratio $\left|\frac{C}{A}\right|^2$

$$(1)+(2) \qquad B = F + G - A$$

$$ik(A-B) = \alpha(G-F) \Rightarrow ik(2A-F-G) = \alpha(G-F)$$

$$2ikA = (\alpha+ik)G - (\alpha-ik)F$$

$$(3)+(4) \qquad -\alpha F e^{-\alpha L} + \alpha G e^{\alpha L} = ik(Fe^{-\alpha L} + Ge^{\alpha L})$$

$$(\alpha-ik)Ge^{\alpha L} = (\alpha+ik)Fe^{-\alpha L}$$

$$G = \frac{\alpha+ik}{\alpha-ik}e^{-2\alpha L}F \qquad \text{substitute}$$

$$2ikA = \frac{(\alpha+ik)^2}{\alpha-ik}e^{-2\alpha L}F - (\alpha-ik)F$$

$$F = \frac{2ik(\alpha-ik)}{(\alpha+ik)^2e^{-2\alpha L} - (\alpha-ik)^2}A \; ; \quad G = \frac{2ik(\alpha+ik)}{(\alpha+ik)^2e^{-2\alpha L} - (\alpha-ik)^2}A$$

$$(3): \quad Ce^{ikL} = Fe^{-\alpha L} + Ge^{\alpha L} = \frac{2ikA}{(\alpha+ik)e^{-2\alpha L} - (\alpha-ik)^2}\left[(\alpha-ik)e^{-\alpha L} + (\alpha+ik)e^{-\alpha L}\right]$$

$$= \frac{4ik\alpha e^{-\alpha L}}{(\alpha+ik)^2e^{-2\alpha L} - (\alpha-ik)^2}A = \frac{4ik\alpha A}{(\alpha+ik)^2e^{-\alpha L} - (\alpha-ik)^2e^{\alpha L}}$$

Denominator: $(\alpha+ik)^2e^{-\alpha L} - (\alpha-ik)^2e^{\alpha L} =$

$$(\alpha^2+2ik\alpha-k^2)e^{-\alpha L} - (\alpha^2-2ik\alpha-k^2)e^{\alpha L} = \underbrace{(\alpha^2-k^2)(e^{-\alpha L}-e^{\alpha L})}_{\text{real part}} + \underbrace{2ik\alpha(e^{\alpha L}+e^{-\alpha L})}_{\text{imaginary part}}$$

$$T = \left|\frac{C}{A}\right|^2 = \left|\frac{Ce^{ikL}}{A}\right|^2 = \frac{16k^2\alpha^2}{(\alpha^2-k^2)^2(e^{-\alpha L}-e^{\alpha L})^2 + 4k^2\alpha^2(e^{\alpha L}+e^{-\alpha L})^2} =$$

$$= \frac{16k^2\alpha^2}{(\alpha^2-k^2)^2(e^{-2\alpha L}-2+e^{2\alpha L}) + 4k^2\alpha^2(e^{2\alpha L}+2+e^{-2\alpha L}-4+4)} =$$

$$= \frac{16k^2\alpha^2}{(\alpha^2+k^2)^2(e^{-\alpha L}-e^{\alpha L})^2 + 16k^2\alpha^2}$$

$$k^2 = \frac{2mE}{\hbar^2} \qquad \alpha^2 = \frac{2m(U_0-E)}{\hbar^2} \qquad k^2+\alpha^2 = \frac{2mU_0}{\hbar^2}$$

$$e^{\alpha L} - e^{-\alpha L} = 2\sinh(\alpha L)$$

$$T = \frac{4k^2\alpha^2}{(k^2+\alpha^2)^2\sinh^2(\alpha L) + 4k^2\alpha^2} = \frac{1}{\frac{(k^2+\alpha^2)^2}{4k^2\alpha^2}\sinh^2(\alpha L) + 1}$$

$$T = \frac{1}{1 + \frac{U_0^2}{4E(U_0 - E)} \sinh^2(\alpha L)}$$

if $\alpha L \gg 1$ (thick or high barrier)

$$\sinh(\alpha L) = \frac{1}{2}\left(e^{\alpha L} - e^{-\alpha L}\right) \approx \frac{1}{2} e^{\alpha L}$$

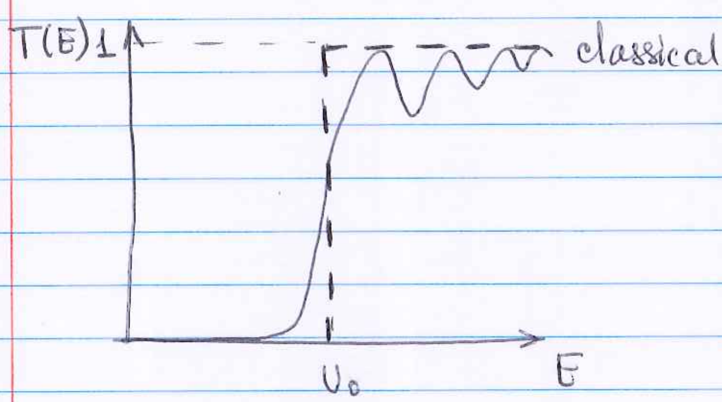
and $\underbrace{1}_{\text{small}} + \frac{U_0^2}{4E(U_0-E)}\left(\frac{1}{2}e^{\alpha L}\right)^2 \approx \frac{U_0^2}{16E(U_0-E)} e^{2\alpha L}$

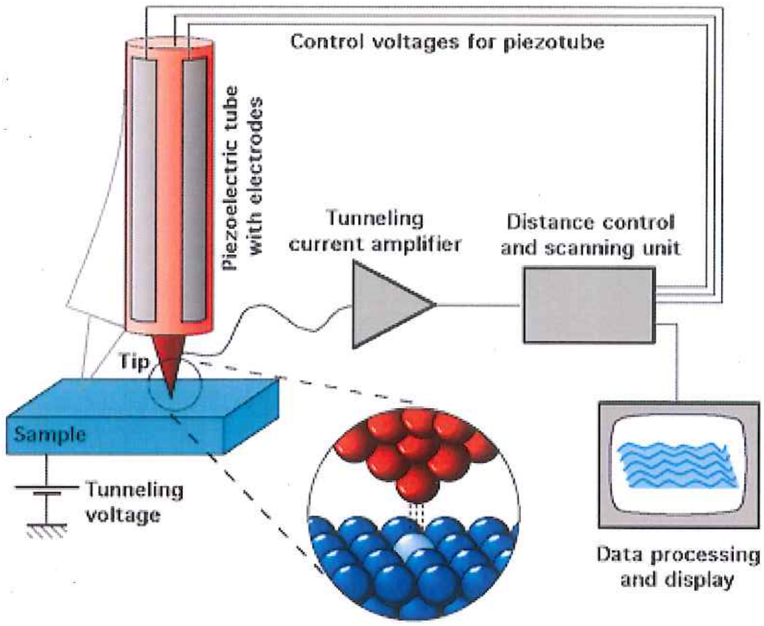
thus, for $\alpha L \gg 1$ $\quad T \approx \frac{16E(U_0-E)}{U_0^2} e^{-2\alpha L}$

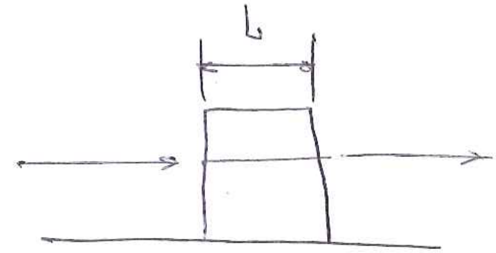
exponentially low transmission probability

Weak (thin) barrier $\quad \alpha L \ll 1 \quad \sinh \alpha L \approx \alpha L$

$$T = \frac{1}{1 + \frac{U_0^2}{4(U_0-E)E}\alpha^2 L^2} = \frac{1}{1 + \frac{U_0^2}{4E\cancel{(U_0-E)}} \frac{2m\cancel{(U_0-E)}}{\hbar^2} L^2} = \frac{1}{1 + \frac{mU_0^2 L^2}{2E\hbar^2}}$$

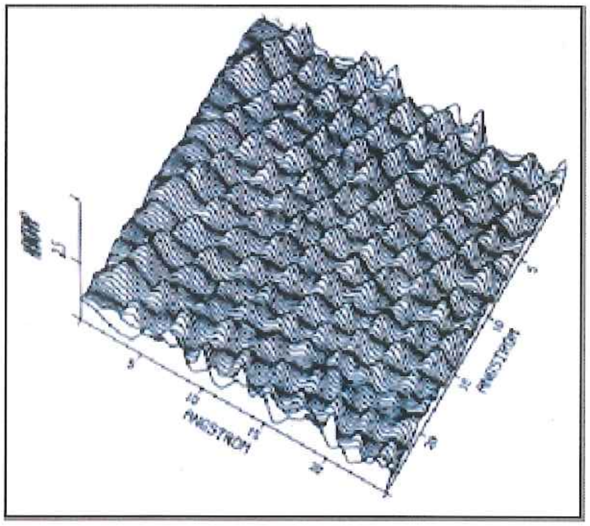






$$P_{tunnel} \simeq e^{-2\alpha L}$$

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$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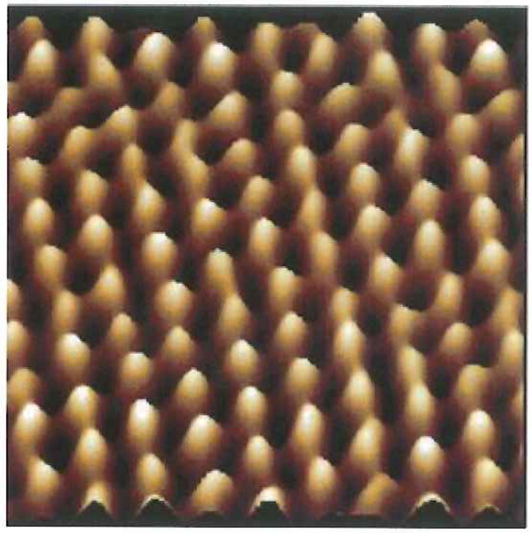

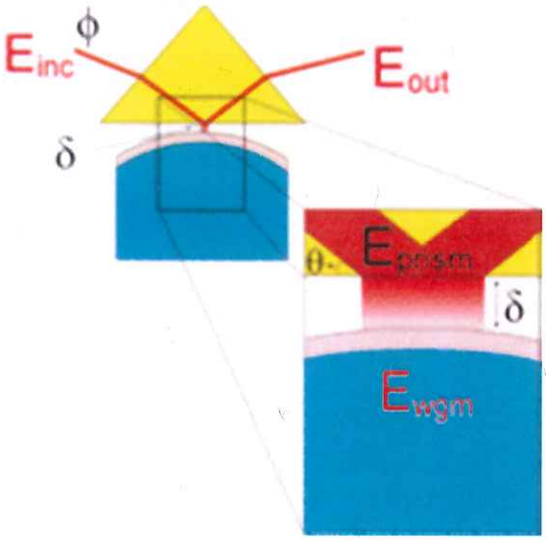

ϕ
E_{inc}
E_{out}
δ
E_{prism}
δ
E_{wgm}