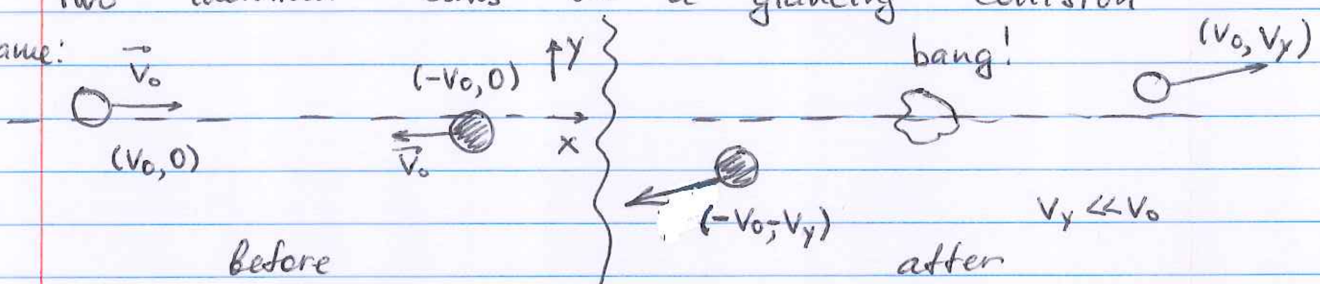
Relativistic energy and momentum

Reminder: all laws of physics must hold in any inertial RF.
That includes the conservation of momentum

Two identical balls in a glancing collision

Lab frame:



If $V_0 = 0.6c$ ($\gamma = 1.25$), and V_y is not relativistic ($V_y \ll c$)
Total momentum before the collision

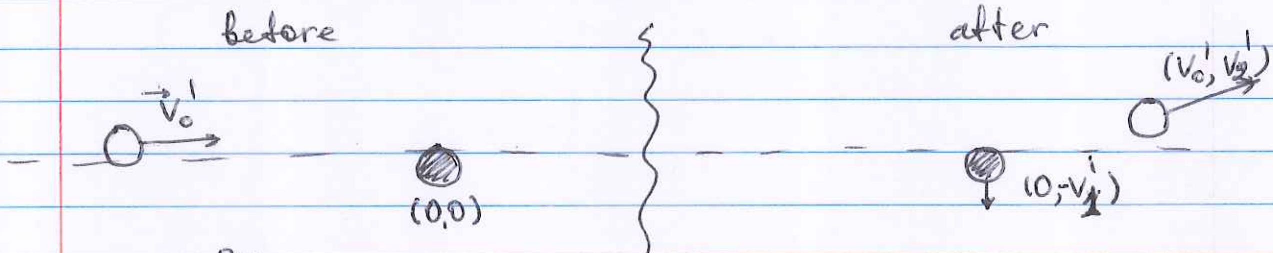
$m(V_0, 0) + m(-V_0, 0) = 0$

After the collision

$m(V_0, V_y) + m(-V_0, -V_y) = 0$

momentum is conserved

Reference frame of the black ball



$$V_0' = \frac{2V_0}{1 + V_0^2/c^2} \approx 0.9c$$

Total momentum before the collision

$m(V_0', 0) + m(0, 0) = m(V_0', 0)$

after the collision

$m(V_0', V_2') + m(0, -V_1') = m(V_0', V_2' - V_1')$

$V_2' \neq V_1'$, even though both V_2' and V_1' are non-relativistic.

White ball moves at $V \approx 0.9c \rightarrow$ its time is dilated, but the black ball is non-relativistic (in this RF)

If we account for the time dilation of the white ball, we'd find

$$v_2' = v_1'/\gamma', \text{ where } \gamma' = \frac{1}{\sqrt{1-(v_x'/c)^2}}$$

so $\vec{P}_{before} \neq \vec{P}_{after}$. Contradiction?

Einstein suggestion: mass of an object depends on its velocity

$$m_v = \frac{m_0}{\sqrt{1-v^2/c^2}} = \gamma m_0$$

m_0 - rest mass (mass measured at its rest reference frame)

Momentum is still $\vec{p} = m_v \cdot \vec{v} = \frac{m_0 \vec{v}}{\sqrt{1-v^2/c^2}} = \gamma m_0 \vec{v}$

What about the energy?

Second Newton law $\vec{F} = \frac{d\vec{p}}{dt}$

Power $\frac{dE_k}{dt} = \vec{F}\cdot\vec{v}$ E - ~~total~~ kinetic energy of an objet

$$\frac{dE_k}{dt} = \vec{v}\,\frac{d(m_v\vec{v})}{dt} = v\frac{d}{dt}\left(m_0\frac{v}{\sqrt{1-v^2/c^2}}\right) =$$

$$= m_0 v\frac{dv}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}\right) = \frac{m_0 v}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt}$$

$$\frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right)$$

if the kinetic energy is $E_k = 0$ at $t=0$

$$E_k(t) = \int_0^t \frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right) = \left.\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right|_{t=0,\ v=0}^{t,\ v} =$$

$$= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$

Kinetic energy E_k (or K) $= \dfrac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2$

Total energy $E = \dfrac{m_0 c^2}{\sqrt{1-v^2/c^2}}$

Rest energy $E_0 = m_0 c^2$ (total energy at $v=0$)

Momentum $\vec{p} = \dfrac{m_0 \vec{v}}{\sqrt{1-v^2/c^2}}$

Are these definitions consistent with the low-energy (familiar) ones?

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \quad \text{for } v/c \ll 1$$

Momentum $\vec{p} = \dfrac{m_0\vec{v}}{\sqrt{1-v^2/c^2}} \xrightarrow{\frac{v}{c} \ll 1} m_0\vec{v}\,(1 + \overbrace{\tfrac{1}{2}\tfrac{v^2}{c^2}}^{\text{relativistic correction}}) \approx m_0\vec{v}$

Kinetic energy $K = \dfrac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2 \xrightarrow{v/c \ll 1}$

$$m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - m_0 c^2 = \frac{1}{2} m_0 \cancel{c^2} \cdot \frac{v^2}{\cancel{c^2}} = \underbrace{\frac{1}{2} m_0 v^2}_{\text{non-relativistic kinetic energy}}$$

In this section one <u>must</u> use the relativistic expression for the kinetic energy, when solving problems!

<u>Big new idea:</u> mass is a form of energy!

Any massive object has its rest energy, that can be used up and converted into other forms of energy.

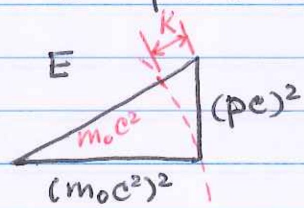
Useful relations b/w energy and momentum

$$p = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \qquad E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}}$$

Note, that E, pc and m_0c^2 all have the units of energy!

① $\frac{p}{E} = \frac{v}{c^2}$ or $\frac{pc}{E} = \frac{v}{c}$

② $E^2 = p^2c^2 + m_0^2 c^4$ (Pythagorian relation)



$E = m_0c^2 + K$ (energy conservation)

"Natural" particle units for energy

electron · Volt or $eV = 1.6 \cdot 10^{-19}\,C \cdot 1V = 1.6 \cdot 10^{-19}\,J$

It is natural since relativistic particles are obtained in accelerators ~~[illegible]~~ where they are accelerated by high voltage.

Mega eV $= 10^6$ eV $=$ 1 MeV

Giga eV $= 10^9$ eV $=$ 1 GeV

Momentum "natural unit" $[pc]$ – unit of energy

$$[p] = \text{MeV}/c \text{ or } \text{GeV}/c$$

Mass "natural unit" $[mc^2]$ – unit of energy

$$[m] = \text{MeV}/c^2 \text{ or } \text{GeV}/c^2$$

Example

Calculate energy, momentum and velocity of a 2-GeV proton

In this ~~to~~ notation, normally the kinetic energy is given,

$K = 2\ \text{GeV}$

proton $\longrightarrow$ we know its mass $\boxed{1\ \text{MeV} = 1.6 \cdot 10^{-13}\ \text{J}}$

$m_p = 1.67 \cdot 10^{-27}\ \text{kg}$

$m_p c^2 = 1.67 \cdot 10^{-27}\ \text{kg} \cdot (3 \cdot 10^8\ \text{m/s})^2 = 1.5 \cdot 10^{-10}\ \text{J} = 938\ \text{MeV}$

$m_p = 938\ \text{MeV}/c^2$ (from the front cover of the book)

Total energy $E = m_p c^2 + K = 0.938\ \text{GeV} + 2\ \text{GeV} = \underline{2.938\ \text{GeV}}$

Since we know E and $m_p c^2$, it is convenient to use the Pythagorian relation to find momentum

$$E^2 = (pc)^2 + (m_p c^2)^2 \Rightarrow pc = \sqrt{E^2 - (m_p c^2)^2} = 2.78\ \text{GeV}$$

$$\underline{p = 2.78\ \text{GeV}/c}$$

Velocity: $\dfrac{pc}{E} = \dfrac{v}{c} = \dfrac{2.78\ \text{GeV}}{2.938\ \text{GeV}} \approx 0.95$

$$\underline{v = 0.95c}$$