Measurements in quantum mechanics

Let's suppose that we know a state of a particle, that is described by a wave function $\psi(x)$.
How now we can predict possible outcome of the experiment? And how we even describe the process of measurements?

We will have to use operators, essentially to describe what operations must be done with the wave function. Notation: $\hat{A}$
The operators we need:

position $\hat{x}$ $\quad \hat{x}\psi(x) = x \cdot \psi(x)$

momentum $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$ $\quad \hat{p}_x\psi = -i\hbar \frac{\partial \psi}{\partial x}$

kinetic energy $\hat{K} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$$\hat{K}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

Energy operator — Hamiltonian $\hat{H} = \hat{K} + \hat{U}$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$$

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi(x)$$

The action of an operator can change the wave function. Thus, the measurement of the associated value (observable) can change the state of the system.

However, sometime the action of the operator does not change the wave function (state), Then, if $\hat{A}\psi = a\psi$, where a is a number, the only outcome of the measurement associated with the operator A ~~is~~ will produce the outcome a.

So for example, if we know that the particle must have a uniquely defined energy E, we must find a state, such that the energy operator, acting on the state, produces $E \cdot$ times the same wave function

$$\hat{H}\psi = E\psi \qquad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi = E\psi$$

That is how we set up Schrodinger equation!

Let's get back to the momentum.
We have discussed how a free particle ($U(x)=0$) can be mathematically described by either complex exponents or sin/cos functions. However, if we now look at the momentum, associated with each state, we will find the difference.

Bound state: the particle moves back and forth, so it may have two values of momentum p_x — positive and negative. On the other hand, an unbound particle moving freely along x-axis will have positive value of p_x, and the one moving in the opposite direction — a negative value.

Bound state $\psi(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x n}{L}$

$$\hat{p}_x \psi(x) = \left(-i\hbar \frac{\partial}{\partial x}\right)\left(\sqrt{\frac{2}{L}} \sin \frac{\pi x n}{L}\right) = -i\hbar \sqrt{\frac{2}{L}}\left(\frac{\partial}{\partial x} \sin \frac{\pi n x}{L}\right) =$$

$$= -i\hbar \sqrt{\frac{2}{L}} \frac{\pi n}{L} \cos \frac{\pi n x}{L}$$ $\leftarrow$ wave function changed

Average value of the momentum measurement

$$\langle p_x \rangle = \int_0^L \psi^*(x)\, \hat{p}_x\, \psi(x)\, dx = \int_0^L \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L} \left(-i\hbar \frac{\pi n}{L}\right) \sqrt{\frac{2}{L}} \cos \frac{\pi n x}{L} dx$$

$$= -i\hbar \frac{2\pi n}{L^2} \int_0^L \sin \frac{\pi n x}{L} \cos \frac{\pi n x}{L} dx = -i\hbar \frac{\pi n}{L^2} \int_0^L \sin \frac{2\pi n x}{L} dx = 0$$

Again, it make sense – since the particle has equal chances to be moving in either direction, the average value of $p_x = 0$.

However, if $\psi(x) = e^{ikx}$ then

$$\hat{p}_x \psi(x) = -i\hbar \frac{\partial}{\partial x} e^{ikx} = (-i\hbar)(ik)\, e^{ikx} = \hbar k\, e^{ikx}$$

$$\hat{p}_x\, \psi(x) = \hbar k\, \psi(x)$$

which means that if a particle has the wave function e^{ikx}, its momentum $p_x = \hbar k$

Similarly, if the wave function is $\psi(x) = e^{-ikx}$

$$\hat{p}_x\, \psi(x) = -i\hbar \frac{\partial}{\partial x} e^{-ikx} = (-i\hbar)(-ik)\, e^{-ikx} = (-\hbar k)\, e^{ikx} = (-\hbar k)\psi(x)$$

then the measurement of the momentum only yeilds $p_x = -kx$.

Thus, we can identify the wave function forms that correspond to distinct particle motion

bound state: wave function is a standing wave

$$\psi(x) = A\cos kx + B\sin kx$$

particle moving ~~in~~ along x-axis

$p_x = \hbar k$

$$\psi(x) = e^{ikx}$$

particle moving against x-axis

$$\psi(x) = e^{-ikx}$$

We can use this knowledge to interpret the behavior of the particle

example 1: $\psi_1(x) = e^{ikx}$ – particle travels along x-axis

example 2: $\psi_2(x) = A e^{ikx} + B e^{-ikx}$ – particle can be found moving in $+x$ derection with the probability $|A|^2$, and in $-x$ direction with the probability $|B|^2$