Angular momentum

Constant linear motion — constant momentum (or kinetic energy), quantized for bound states

Constant rotation $\longrightarrow$ constant angular momentum
$$\vec{L} = \vec{r} \times \vec{p} \quad \text{or} \quad I \cdot \vec{\omega} \quad \Rightarrow \quad \text{Quantum} \quad \hat{L} = -i\hbar \vec{r} \times \vec{\nabla}$$

$$\hat{L}_x = \cancel{\quad} \quad y \cdot p_z - z \cdot p_y \quad \Rightarrow \hat{L}_x \psi = -i\hbar \left(y \frac{\partial \psi}{\partial z} - z \frac{\partial \psi}{\partial y} \right)$$
$$\hat{L}_y = z p_x - x p_z \qquad L_z = x p_y - y p_z$$

Because of the uncertainty principle, one cannot determine all three $\vec{L}$ components with arbitrary precision, only 2 of them.

However, in spherically symmetric potential we can use total angular momentum L^2 and L_z to describe the state of the particle. The reason is because spherical functions $Y_{\ell m}(\theta, \varphi)$ are eigenfunctions of these two operators

$$\hat{L}^2 Y_{\ell m}(\theta, \varphi) = -\hbar^2 \left(\frac{\partial^2 Y_{\ell m}}{\partial \theta^2} + \cot\theta \frac{\partial Y_{\ell m}}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{\ell m}}{\partial \varphi^2} \right) = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$\hat{L}^2 Y_{\ell m}(\theta, \varphi) = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

and
$$\hat{L}_z Y_{\ell m} = -i\hbar \frac{\partial Y_{\ell m}}{\partial \varphi} = \hbar m \, Y_{\ell m}$$

So if an electron is at the quantum state
$$\psi_{n \ell m}(r, \theta, \varphi) = R_{n\ell}(r) \, Y_{\ell m}(\theta, \varphi)$$
we know that its measured total angular momentum $\langle L^2 \rangle = \hbar^2 \ell(\ell+1)$ or $\langle |\vec{L}| \rangle = \sqrt{\langle L^2 \rangle} = \hbar \sqrt{\ell(\ell+1)}$ and at the same time $\langle L_z \rangle = \hbar m$ and of course its energy will be $E_n = -\frac{E_R}{n^2}$

Angular momentum

We found that in a hydrogen atoms each stationary quantum state is characterized by three quantum numbers

- principle quantum number $n \rightarrow$ defines energy

$$E_n = - \frac{E_R}{n^2} \qquad n = 1, 2, \ldots$$

- orbital quantum number $\ell \rightarrow$ defines value of angular momentum

$$\langle L^2 \rangle = \hbar^2 \ell(\ell+1) \qquad \ell = \underbrace{0, 1, \ldots n-1}_{n \text{ states degeneracy}}$$

- azimuthal quantum number $m \rightarrow$ defines value of z-component of L_z

$$\langle L_z \rangle = \hbar m \qquad m = \underbrace{0, \pm 1, \ldots \pm \ell}_{2\ell+1 \text{ states degeneracy}}$$

In quantum mechanics we cannot determine all three components of the angular momentum. In spherically symmetric cases we typically determine $\langle L^2 \rangle$ and $\langle L_z \rangle$
Also, the values for both are quantized, so only particular directions are possible

$$\langle |L| \rangle = \sqrt{\langle L^2 \rangle} = \hbar \sqrt{\ell(\ell+1)}$$
$$\ell = 1 \qquad \langle |L| \rangle = \sqrt{2}\,\hbar$$
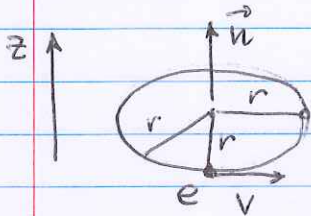$$\ell = 2 \qquad \langle |L| \rangle = \sqrt{6}\,\hbar$$
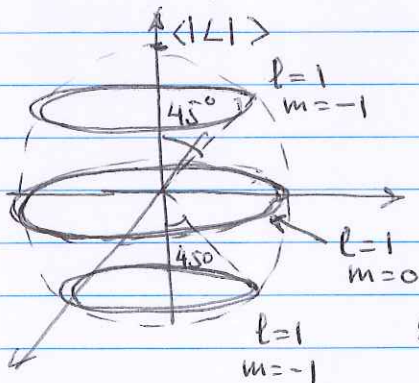
For each $\langle |L| \rangle = \hbar \sqrt{\ell(\ell+1)}$ only specific values of L_z are possible

$$\cos\theta = \frac{\langle L_z \rangle}{\langle |L| \rangle} = \frac{\hbar m}{\hbar \sqrt{\ell(\ell+1)}} = \frac{m}{\sqrt{\ell(\ell+1)}} < 1$$

$\ell = 1 \qquad \cos\theta = \pm 1/\sqrt{2} \quad$ for $\quad m = \pm 1$
$\qquad\qquad \cos\theta = 0 \qquad\quad$ for $\quad m = 0$

Why L_z is physically significant?
It defines the interaction with magnetic field

Classical atom: an electron moving in circle
$\#$ => electric current $I = (-e)/_T = (-e)\frac{v}{2\pi r}$
Magnetic moment $\vec{\mu} = I \cdot \vec{A} = \frac{(-e) \cdot v}{2\pi r} \cdot \pi r^2 \, \vec{n} =$

$$= -\frac{1}{2} e r v \, \vec{n} = -\frac{e}{2m_e} \vec{L}$$

This relationship b/w the magnetic and orbital angular momentum remains in quantum mechanics as well
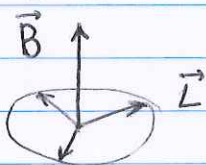
$$\hat{\vec{\mu}} = -\frac{e}{2m_e} \hat{\vec{L}}$$

If a magnetic field $\vec{B} = B\vec{e}_z$ is applied along the z-direction, then the electron's potential energy changes $\hat{U}_{magn} = -\hat{\vec{\mu}} \cdot \vec{B} =$

$$= -\mu_z B_z = \frac{e}{2m_e} \hat{L}_z \cdot B_z$$

Thus for an electron in one of the hydrogen stationary states we can calculate the energy shift in magnetic field

$$\hat{U}_{magn} \, \Psi_{n\ell m} = \frac{e}{2m_e} B_z \, \hat{L}_z \, \Psi_{n\ell m} = \frac{e}{2m_e} B_z \cdot \hbar m$$
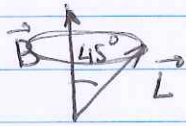
$\ell = 1 \qquad m = 0$

$\vec{B} \uparrow$

$\vec{B} \cdot \vec{L} = 0 \qquad U_{magn} \Psi_{n10} = 0$
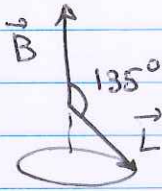
no change in energy

$l = 0 \qquad m = +1$

$$\hat{U}_B \, \psi_{n21} = \frac{\hbar e}{2m_e} \cdot B_z$$

The ~~sta~~ electron ~~acqu~~ ~~acqu~~ acquires
additional energy, proportional to B_z

$l = 0 \qquad m = -1$
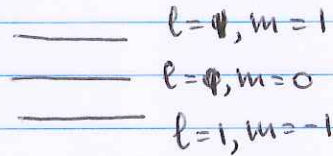
$$\hat{U}_B \, \psi_{n1-1} = -\frac{\hbar e}{2m_e} \cdot B_z$$

The electron energy goes down
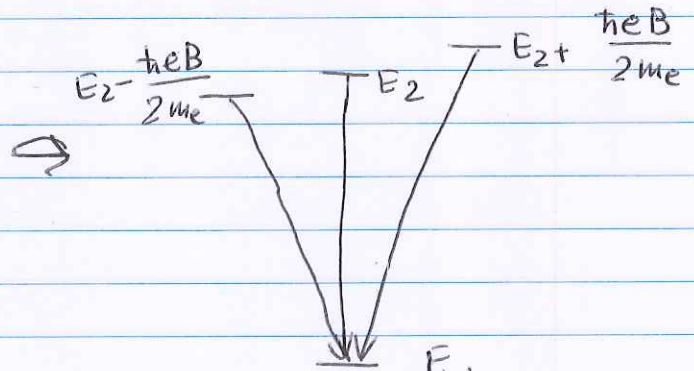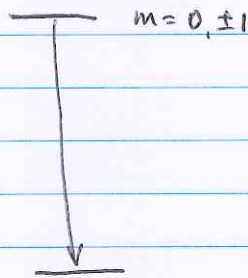with increasing magnetic field

No magnetic field $\qquad\qquad\longrightarrow\qquad$ Applied magnetic field
$\quad l = 1 \to$ 3-fold degeneracy $\qquad\qquad$ degeneracy lifted

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ normal

$E_n \; \dfrac{l=1}{m = 0, \pm 1} \Longrightarrow$
$\qquad\qquad\qquad$ ——— $l = 1, m = 1 \qquad$ Zeeman
$\qquad\qquad\qquad$ ——— $l = 1, m = 0 \qquad$ effect
$\qquad\qquad\qquad$ ——— $l = 1, m = -1$

$n = 2, l = 1$ ——— $m = 0, \pm 1 \qquad\qquad\qquad\qquad\qquad$ $E_2 + \dfrac{\hbar e B}{2m_e}$

$\qquad\qquad\qquad\qquad\qquad\qquad E_2 - \dfrac{\hbar e B}{2m_e} \qquad E_2$

$n = 1, l = 0$ ———

transition $\qquad \hbar\omega = E_2 - E_1 = \dfrac{3E_R}{4} \qquad\qquad \hbar\omega = \dfrac{3E_R}{4}, \; \dfrac{3E_R}{4} \pm \dfrac{\hbar e B}{2m_e}$
frequency $\qquad\qquad$ single frequency $\qquad\qquad\qquad\qquad$ 3 frequencies

Because of the selection rules $(\Delta l = 1, \Delta m = 0, \pm 1)$
there will be only 3 spectral lines
for any value of l
$\qquad$ Normal Zeeman effect