Important mathematical identities involving complex exponents

$e^{ikx} = \cos kx + i\sin kx$ $\qquad$ $\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx})$

$e^{-ikx} = \cos kx - i\sin kx$ $\qquad$ $\sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx})$

$|e^{ikx}|^2 = e^{ikx} \cdot e^{-ikx} = 1$

Solutions of the Schrodinger equation for constant potential energy

a) $U = 0$ $\qquad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} + \overbrace{\frac{2mE}{\hbar^2}}^{k^2}\psi = 0 \Rightarrow \psi'' + k^2\psi = 0$

General solution $\qquad \psi(x) = A\cos kx + B\sin kx$ $\qquad k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{2\pi}{\lambda}$

or, alternatively $\qquad \psi(x) = C_1 e^{ikx} + C_2 e^{-ikx}$

Mathematically, the both forms are equivalent, but we usually use the first form for bound states (standing waves), and the second form for the unbound particles, as they let us easily associate ~~interprete~~ two complex exponents with the particles moving along and against x-axis.

b) $U = U_0$ and $E > U_0$ $\qquad$ classically allowed region

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} + \overbrace{\frac{2m(E-U_0)}{\hbar^2}}^{k_1^2}\psi = 0 \Rightarrow \psi'' + k_1^2\psi = 0$

General solution $\qquad \psi(x) = A\cos k_1x + B\cos k_1x$ $\qquad k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}} = \frac{2\pi}{\lambda_1}$

or $\qquad \psi(x) = C_1 e^{ik_1x} + B e^{-ik_1x}$

c) $U = U_0$ and $E < U_0$ $\qquad$ classically forbidden region

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} - \overbrace{\frac{2m(U_0-E)}{\hbar^2}}^{\alpha^2}\psi = 0 \Rightarrow \psi'' - \alpha^2\psi = 0$

General solution: $\qquad \psi(x) = Ce^{\alpha x} + De^{-\alpha x}$ $\qquad \alpha = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$