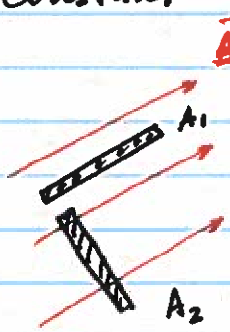
Gauss law for electric field

Electric flux

If we imagine an electric charge to be either an emitter (+) or a sink (−) of electric field (lines), then we can define the flux as the number of field lines crossing a certain area

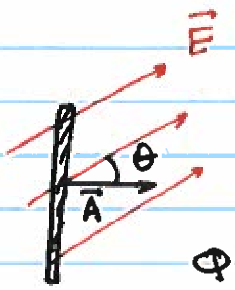
Constant $\vec{E}$ - field



Flux through A_1 is zero (A_1 is parallel to $\vec{E}$)

Flux through A_2 is maximal $\Phi = \vec{E} \cdot \vec{A}_2$

In general $\Phi = \vec{E} \cdot \vec{A} = |\vec{E}| \underbrace{|\vec{A}| \cos\theta}_{A_\perp} = E \cdot A_\perp$



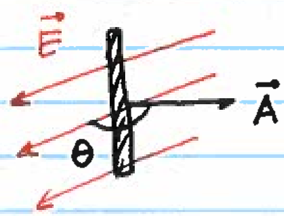
$\theta < 90°$

$\vec{A}$ – vector area

$|\vec{A}|$ = area of the surface

direction – perpendicular to the surface

$\Phi > 0$



$\theta > 90°$

$\Phi < 0$

If $\vec{E}$ (or $\vec{A}$) is changing in space: $d\Phi = \vec{E}(\vec{r})\, d\vec{A}$

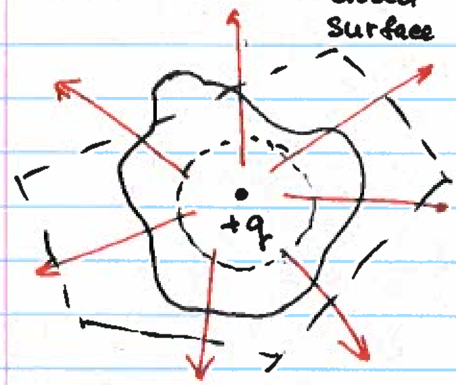
$$\Phi = \int \vec{E}(\vec{r})\, d\vec{A} \qquad \text{surface integral}$$

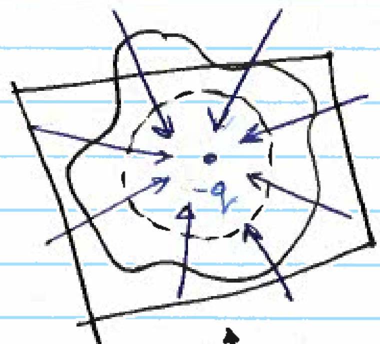
Gauss Law

The flux of electric field $\vec{E}$ through one <u>closed</u> surface is equal to the net charge enclosed q_{enc}, divided by ε_0.

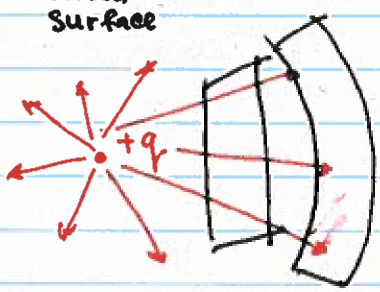
$$\Phi_{\substack{\text{closed}\\ \text{surface}}} = \oint \vec{E}\, d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \qquad \left[k = \frac{1}{4\pi\varepsilon_0}\right]$$

2D





$$\Phi_{\substack{\text{any}\\ \text{closed}\\ \text{surface}}} = q/\varepsilon_0 > 0 \qquad\qquad \Phi_{\substack{\text{any}\\ \text{closed}\\ \text{surface}}} = q/\varepsilon_0 < 0$$

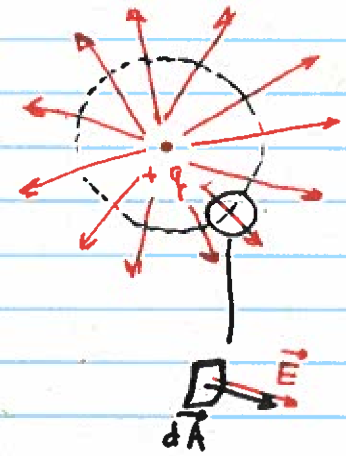


$\leftarrow \Phi = 0$

Gauss law is always valid, and is the first of four Maxwell's equations that rules electromagnetic world.

However, in some situations we can use it to calculate the electric field, if we take into account some symmetries of the problem.

In fact we can derive the Coulumb law from the Gauss law



Due to the symmetry, $\vec{E}$ must be the same ~~at~~ magnitude and point radially at the same distance r.

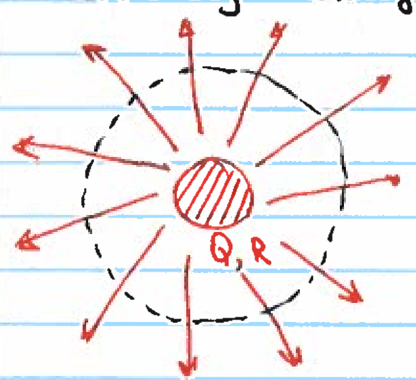
Both $\vec{E}$ and $d\vec{A}$ are along the radius

$$d\Phi = \vec{E} \cdot d\vec{A} = E \cdot dA$$

$$\Phi = \underbrace{\int d\Phi}_{\text{sphere of radius } r} = \int E\,dA = E \int dA = E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\boxed{E(r) = \frac{q}{4\pi\varepsilon_0 \cdot r^2}} \qquad \vec{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$

Uniformly charged sphere



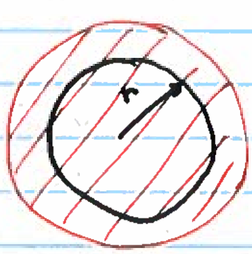
If outside the sphere $\rightarrow$ identical to a point charge!

$$\Phi = E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{r} \qquad r > R$$

Inside the sphere the charges are uniformly distributed

Volume charge density $\rho = \dfrac{Q}{\frac{4\pi}{3}R^3}$



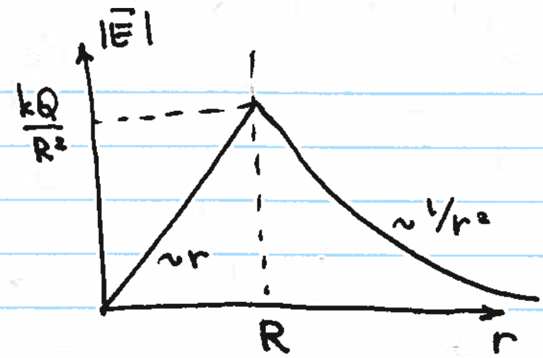
$$q_{enc} = \rho \cdot \frac{4\pi}{3} r^3 = Q\,\frac{r^3}{R^3}$$

$$\Phi = E(r) \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} Q\,\frac{r^3}{R^3}$$

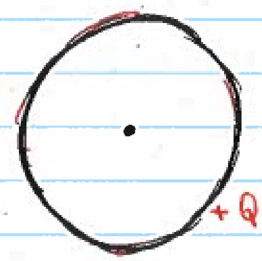
vector $\vec{r}$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q\vec{r}}{R^3} \qquad r<R$$

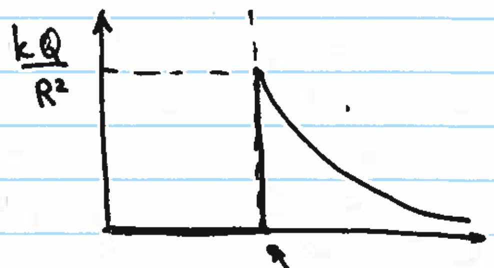
$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0} Q/r^2 \, \hat{r} & r>R \\ \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3} \vec{r} & r<R \end{cases}$$



Thin charged shell



$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} & r>R \\ 0 & r<R \end{cases}$$



this discontinuity indicate the location of the charge

We can use such found electric field to find corresponding electrostatic potential.
If $\vec{E}$ is radial, then $E_r = -\frac{dV}{dr}$

and $V = -\int^{r} E_r \, dr'$

Point charge or outside of charged sphere

$$\vec{E} = \frac{kq}{r^2}\hat{r} \qquad V(r) = -\int^{r} \frac{kq}{r'^2} dr' = \frac{kq}{r} + \text{const}$$

our choice

Uniformly charged sphere

Inside $\quad E_r = \frac{kQ}{R^3} r \qquad V(r) = \int E_r dr = \frac{kQ}{R^3} \int r dr =$

$$= -\frac{kQ r^2}{2R^3} + \text{const}$$

To make the potential continuous $\quad -\frac{kQr^2}{2R^3} + \text{const} \Big|_{r=R} = \frac{kQ}{R}$

$$\text{const} = \frac{3kq}{2R}$$

$$V(r) = \begin{cases} \frac{kQ}{r} & r > R \\ \frac{kQ}{2R}\left[3 - \frac{r^2}{R^2}\right] & r < R \end{cases}$$

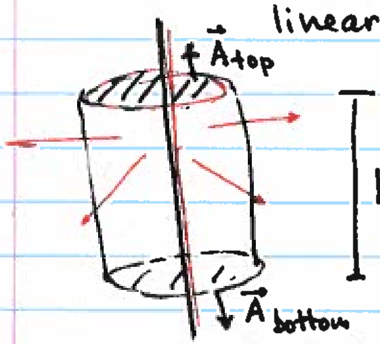
Hollow charged shell

$$V(r) = \begin{cases} \frac{kQ}{r} & r > R \\ \frac{kQ}{R} & r < R \end{cases}$$

$\leftarrow$ constant, since $E = 0$ inside

Infinite charged wire

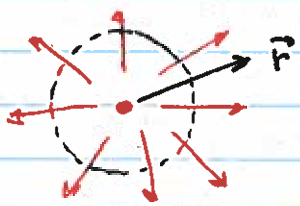
linear charge density λ -



$$\Phi_{cylinder} = \underbrace{\Phi_{top} + \Phi_{bottom}}_{=0 \quad \text{since } \vec{A} \perp \vec{E}} + \Phi_{side}$$

$$\Phi_{side} = E(r) \cdot 2\pi r \cdot \cancel{L} = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda \cdot \cancel{L}}{\varepsilon_0}$$

top view

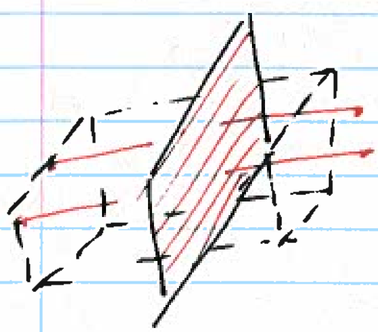


$$\vec{E}(\vec{r}) = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} \hat{r}$$

Are dimentions ok? Yes

$\lambda = [Q]/[r] \qquad [\lambda/r] = [Q]/[r^2]$ ✓

Infinite charged plane



surface Charge σ

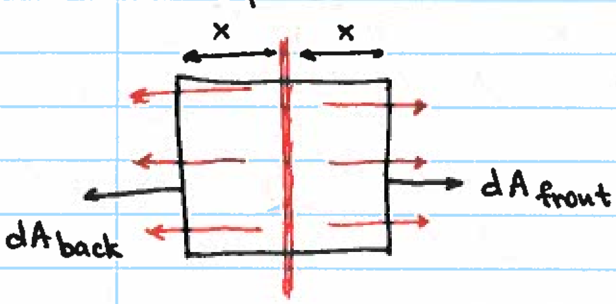
$$\Phi = \Phi_{sides} + \Phi_{front} + \Phi_{back}$$

$$\Phi_{front} = A_{front} \cdot E(x)$$

$$\Phi_{back} = A_{back} \cdot E(x)$$

$$\Phi_{sides} = 0$$

top view

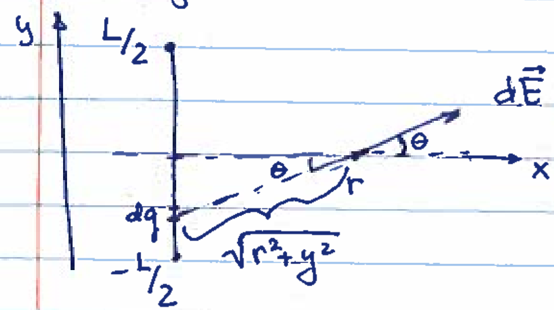


$$\Phi = 2\cancel{A} \cdot E(x) = \cancel{A} \cdot \sigma \frac{1}{\varepsilon_0}$$

$$E = \frac{1}{2\varepsilon_0} \cdot \sigma$$

does not depend on the distance from the surface (but we already know that!)

When we can use Gauss law?
An infinite wire really doesn't exist, so is it a useless exercize? Not, if the answer it gives is within accuracy you need.



Due to symmetry only dE_x will contribute to the electric field at the center of the rod

$\lambda = Q/L$

$$dE_x = \frac{k\,dq}{(r^2+y^2)}\underbrace{\frac{r}{\sqrt{r^2+y^2}}}_{\cos\theta} = \frac{rk\lambda\,dy}{(r^2+y^2)^{3/2}}$$

$$E_x = \int_{-L/2}^{L/2} \frac{k\lambda r\,dy}{(r^2+y^2)^{3/2}} = k\lambda r \int_{-L/2}^{L/2} \frac{dy}{(y^2+r^2)^{3/2}} = k\lambda r \left. \frac{y}{r^2\sqrt{y^2+r^2}} \right|_{-L/2}^{L/2}$$

$$= \frac{\lambda k L}{r\sqrt{r^2+L^2/4}}$$

Infinite rode $\quad E_{inf} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} = \frac{2k\lambda}{r}$

If $L\to\infty$ $\quad \lim \frac{\lambda k L}{r\sqrt{r^2+L^2/4}} \to \frac{2\lambda k}{r}$

↑ neglect

$$\frac{E_L}{E_{inf}} = \frac{L/2}{\sqrt{r^2+L^2/4}}$$