3.24) for a vacuum state with
nction

$$+ \frac{\partial}{\partial q}\Big) \varphi_0(q) \tag{3.31}$$

$$\varphi_{-1}(q) = 0. \tag{3.32}$$

$$\frac{q^2}{2}\Big) \tag{3.33}$$

inhomogeneous solution of
nalizable as well. It is called
m state and is given by the

$$\Big) \operatorname{erfi}(q) \tag{3.34}$$

n

$$(t^2)\, dt. \tag{3.35}$$

3.2.3 Thermal states

Most natural light is thermal radiation: sunlight, for example, or the light of most lamps. Thermal radiation is a state of the electromagnetic field in thermal equilibrium (or part of the field in thermal equilibrium, for example the part radiating out of a thermal source). Note that light on its own cannot reach an equilibrium state, because it is not interacting with itself, but when light is brought into contact with material media or generated by thermal sources it may thermalize, as we discuss in Chapter 6 and in particular in Section 6.2.2. In interacting with a hot material the electromagnetic field exchanges photons. So the energy and the photon number of thermal light fluctuates.

Let us derive the quantum state of thermal light without assuming that the reader is familiar with quantum statistical mechanics. In thermal equilibrium light is in a stationary state that does not evolve in time. The most general stationary state is a statistical mixture of the eigenstates of the Hamiltonian $\hat{H}$, a mixture of Fock states,

$$\hat{\rho} = \sum_n \rho_n \, |n\rangle\langle n|. \tag{3.38}$$

In addition to being stationary, thermal light is in a state of maximal disorder, maximal entropy S for a given energy E that is set by the ceaseless exchange of photons with the thermal environment, the hot material. Another natural constraint on ρ_n is the conservation of the total probability: the probabilities ρ_n must sum up to unity. It is mathematically convenient to describe the maximization of the entropy in the presence of constraints using Lagrange multipliers. We simply subtract the constraints from the entropy (1.21) with variable prefactors,

$$S = -k_B \sum_n \rho_n \ln \rho_n - a\Big(\sum_n \rho_n - 1\Big) - b\Big(\sum_n \rho_n E_n - E\Big), \tag{3.39}$$

where E denotes the average energy of the mode. If we optimize the entropy (3.39) for the parameters a and b we demand that the derivatives of S with respect to a and b vanish. In this case, both constraints on the ρ_n follow. On the other hand, when the constraints are satisfied the modified entropy (3.39) agrees with the original (1.21). So the extremum of S with respect to all the ρ_n and the parameters a and b, the Lagrange multipliers, solves the optimization problem with constraints. We obtain by differentiation

$$0 = \frac{\partial S}{\partial \rho_n} = -k_B\left(\ln \rho_n + 1\right) - a - b\,E_n \tag{3.40}$$

that has the solution

$$\rho_n = \frac{1}{Z}\exp\left(-\frac{E_n}{k_B T}\right) \tag{3.41}$$

where T denotes $1/b$ and Z abbreviates $\exp(1 + a/k_B)$. As we will see in a moment, T is the *temperature* while Z is the *partition function* or *statistical sum*, because

$$Z = \sum_n \exp\left(-\frac{E_n}{k_B T}\right) \tag{3.42}$$

in order to satisfy the conservation of the total probability. When thermal equilibrium is reached the entropy assumes the value

$$S = -k_B \sum_n \rho_n \ln \rho_n = k_B \ln Z + \frac{E}{T} \tag{3.43}$$

where E denotes the average energy

$$E = \sum_n \rho_n E_n. \tag{3.44}$$

In thermodynamics, E is the internal energy. Suppose that T varies. We obtain from the definition (3.42) of the statistical sum that

$k_B T^2 \mathrm{d}Z = ZE\,\mathrm{d}T$. Hence we get for the derivative of the entropy (3.43) the expression

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{T}. \tag{3.45}$$

This formula is the definition of the thermodynamic temperature (Landau and Lifshitz, Vol. V, 1996), which justifies our terminology. We thus obtained *Boltzmann's formula* (3.41) for the statistical distribution of states with energies E_n in thermal equilibrium with temperature T.

So far our analysis has been rather general, we have not used the specific physics of the electromagnetic oscillator, but derived the essentials of the quantum statistical mechanics of a *canonical Gibbs ensemble*. We can combine our results (3.38), (3.41) and (3.42) in the compact expressions

$$\hat{\rho} = \frac{1}{Z}\exp\left(-\frac{\hat{H}}{k_B T}\right), \quad Z = \mathrm{tr}\left\{\exp\left(-\frac{\hat{H}}{k_B T}\right)\right\}. \tag{3.46}$$

For the specific case of a single light mode we use the parameter

$$\beta = \frac{\hbar\omega}{k_B T}. \tag{3.47}$$

In statistical physics, the inverse temperature $1/(k_B T)$ is typically denoted by β, but here we have included $\hbar\omega$ in β for convenience. For the electromagnetic oscillator of the light mode, the statistical sum is the geometric series

$$Z = \sum_{n=0}^{\infty} \mathrm{e}^{-n\beta} = \frac{1}{1-\mathrm{e}^{-\beta}}. \tag{3.48}$$

Therefore, the thermal state of light is described by the density matrix

$$\hat{\rho} = \left(1 - \mathrm{e}^{-\beta}\right)\sum_{n=0}^{\infty} \mathrm{e}^{-n\beta}\,|n\rangle\langle n|. \tag{3.49}$$

For the average photon number, we get the *Planck spectrum* of the harmonic oscillator in thermal equilibrium,

$$\bar{n} = \frac{1}{Z}\sum_{n=0}^{\infty} \mathrm{e}^{-n\beta} n = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{1}{\mathrm{e}^{\beta}-1}. \tag{3.50}$$

When the entire electromagnetic field is in a thermal state, we obtain the total energy density $\varrho(\omega)$ per volume and frequency by summing over the individual energies $\hbar\omega\,\bar{n}$ of all the modes with the same frequency ω and dividing by the total volume. In empty space, we arrive at Planck's radiation formula that, historically, opened one of the windows to the quantum world (Jammer, 1989),

$$\varrho(\omega) = \left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)\frac{1}{\exp\left(\hbar\omega/(k_B T)\right) - 1}. \tag{3.51}$$