Thermal states of e-m field

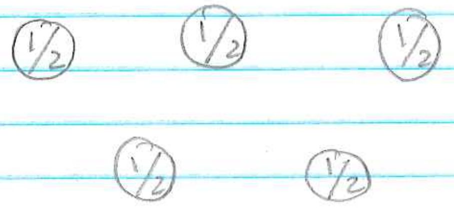
Thermal radiation is a state of e-m field in thermal equilibrium. Light on its own cannot reach the thermal equilibrium as it does not interact with itself, but when light is brought into contact with material media or generated by thermal source it may thermalize. As a result of such interaction photons of different frequencies are absorbed or emitted, and thus the energy and photon number fluctuates, and thermal field must contain e-m components (modes) of different frequency

Pure vs mixed states

1 particle in a quantum superposition

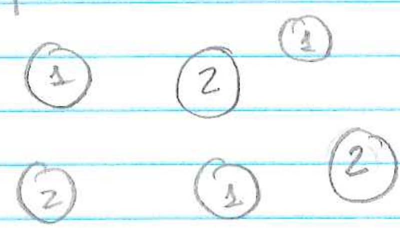
$$\Psi_{1/2} = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

N particles in $\Psi_{1/2}$



When measured, 50% of particles will be found in state $|1\rangle$, but before the measurement <u>all</u> N-particles are in the same state!

Statistical mixture of particles



Some particles are always in $|1\rangle$, and some always in $|2\rangle$

No wave function is defined.

Density matrix operator

$$\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

Here $\{|\psi_n\rangle\}$ are the basis states of the system

Pure state $\quad \psi_{1/2} = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \quad \hat{\rho} = |\psi_{1/2}\rangle\langle\psi_{1/2}|$

$$\hat{\rho}_{pure} = \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) = \frac{1}{2}\left[|1\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right]$$

$$\hat{\rho}_{pure} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

quantum coherence
superposition principle!

Mixed state

$p_1 = 1/2 \qquad p_2 = 1/2$

$$\rho_{mixed} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|$$

$$\rho_{mixed} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The probability of finding a system in a state $|\varphi\rangle$ is given by

$$0 \leq P_\varphi = \langle\varphi|\hat{\rho}|\varphi\rangle \leq 1$$

In oure case the probability of both of our considered systems to be in state $|1\rangle$

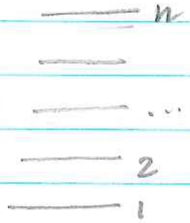
Pure:

$$\langle 1|\hat{\rho}_{pure}|1\rangle = (1\ 0)\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 1/2$$

Mixed

$$\langle 1|\hat{\rho}_{mixed}|1\rangle = (1\ 0)\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = 1/2$$

Thermalized harmonic oscillator



Classical case – Boltzman distribution

$$P_n = \frac{e^{-E_n/k_BT}}{\underbrace{\sum_n e^{-E_n/k_BT}}_{Z}} \qquad E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

Quantum case:

$$\hat{\rho}_{th} = \frac{e^{-\hat{H}/k_BT}}{\underbrace{\mathrm{Tr}\left[e^{-\hat{H}/k_BT}\right]}_{Z}} \qquad \hat{H} = \hbar\omega\left(\hat{n}+\tfrac{1}{2}\right) = \hbar\omega\left(\hat{a}^+\hat{a}+\tfrac{1}{2}\right)$$

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_BT}\right] = \sum_{n=0}^{\infty}\langle n|e^{-\hat{H}/k_BT}|n\rangle =$$

$$= \sum_{n=0}^{\infty} e^{-\hat{E}_n/k_BT} = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+1/2)/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{1-e^{-\hbar\omega/k_BT}}$$

Average number of photons

$$\langle\hat{n}\rangle = \mathrm{Tr}\langle\hat{n}\hat{\rho}_{th}\rangle = \sum_{n=0}^{\infty}\langle n|\hat{n}\hat{\rho}_{th}|n\rangle =$$

$$= \sum_{n=0}^{\infty} n\,\underbrace{\langle n|\hat{\rho}_{th}|n\rangle}_{P_n} = \frac{1}{Z}\sum_{n=0}^{\infty} n\,e^{-E_n/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{Z}\sum_{n=0}^{\infty} n\cdot e^{-\frac{\hbar\omega n}{2k_BT}}$$

$$\sum_{n=0}^{\infty} n e^{-\beta n} = \frac{e^{-\beta}}{(1-e^{-\beta})^2}$$

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} \longrightarrow \begin{cases} \hbar\omega \ll k_BT & \text{(thermal radiation)} \\ \quad \bar{n}\approx k_BT/\hbar\omega & \\ \hbar\omega \gg k_BT & \text{(optical range for room temperature)} \\ \quad \bar{n}\approx e^{-\hbar\omega/k_BT} & \end{cases}$$

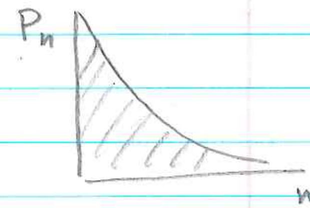
Bose-Einstein statistics

Since $e^{-\frac{\hbar\omega}{k_B T}} = \frac{\bar{n}}{1+\bar{n}}$

we can rewrite

$$\hat{\rho}_{th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$$

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$$



$$\Delta n = \sqrt{\bar{n} + \bar{n}^2}$$

$$\frac{\Delta n}{n} = \sqrt{1 + \frac{1}{\bar{n}}} \xrightarrow{\bar{n} \gg 1} 1$$

$$\xrightarrow{\bar{n} \ll 1} \infty$$

Total energy density

density of states

$$u(\omega) = \hbar\omega \, \bar{n}(\omega) \cdot \rho(\omega)$$

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$ Planck's radiation law

One can use it to derive all the laws describing black body radiation.