Classical electrodynamics

Light is an electromagnetic wave.

We "know" this from Maxwell eqns

no sources

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \nabla \cdot \vec{D} = 0 \end{cases}$$

ε - permittivity
μ - permeability

$\vec{E}$ - electric field
$\vec{D}$ - electric displacement
$\vec{B}$ - magnetic induction
$\vec{H}$ - magnetic field

$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$, $\vec{B} = \mu_0 \mu H$] isotropic medium

$\varepsilon_0 \mu_0 = 1/c^2$

Quantum electrodynamics: $\hat{\vec{E}}, \hat{\vec{D}}, \hat{\vec{B}}, \hat{\vec{H}}$

Each classical field corresponds to an operator for example,

$$\vec{E} = \langle \psi | \hat{\vec{E}} | \psi \rangle$$

Quantum fields must obey Maxwell equations as well! (since they are linear)

$$\nabla \times \langle \psi | \hat{\vec{E}} | \psi \rangle = -\frac{\partial \langle \psi | \hat{\vec{B}} | \psi \rangle}{\partial t}$$

$$\langle \psi | \nabla \times \hat{\vec{E}} + \frac{\partial \hat{\vec{B}}}{\partial t} | \psi \rangle = 0 \quad \text{for } \forall \psi$$

$$\nabla \times \hat{\vec{E}} + \frac{\partial \hat{\vec{B}}}{\partial t} = 0$$

(and so on for other equations)

Similarly $\hat{\vec{D}} = \varepsilon_0 \varepsilon \hat{\vec{E}}$, $\hat{\vec{B}} = \mu_0 \mu \hat{\vec{H}}$

Classical e-m field energy

$$H = \frac{1}{2}\int dV \left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H} \right)$$

Corresponding QED Hamiltonian

$$\hat{H} = \frac{1}{2}\int dV \left(\hat{\vec{E}}\cdot\hat{\vec{D}} + \hat{\vec{B}}\cdot\hat{\vec{H}} \right) = \frac{1}{2}\int dV \left(\varepsilon_0 \varepsilon \hat{\vec{E}}^2(\vec{r},t) + \frac{1}{\mu_0 \mu} \hat{\vec{B}}^2(\vec{r},t) \right)$$

$\hat{\vec{E}}$ and $\hat{\vec{B}}$ are not independent, but connected through Maxwell eqns.

Vector potential $\hat{\vec{A}}$

$\hat{\vec{E}} = -\partial\hat{\vec{A}}/\partial t$ $\quad$ $\hat{\vec{B}} = \nabla \times \vec{A}$

Coulumb gauge: $\nabla(\varepsilon\vec{A}) = 0$ [and so $\nabla\vec{D} = 0$].

wave equation for $\hat{\vec{A}}$

$$\frac{1}{\varepsilon}\nabla\times\frac{1}{\mu}\nabla\times\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

isotropic medium $\varepsilon \neq \varepsilon(\vec{r})$, $\mu \neq \mu(\vec{r})$

$$\nabla^2\hat{\vec{A}} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\hat{\vec{A}}}{\partial t^2} = 0$$

Classical problems:

Typical way to proceed: consider boundary conditions for the problem, find a good mode basis that satisfies these boundary conditions $\vec{A}_k(\vec{r},t)$, and then figure out what combinations of these modes provides the right solution

$$\vec{A}(\vec{r},t) = \sum_k c_k \vec{A}_k(\vec{r},t)$$

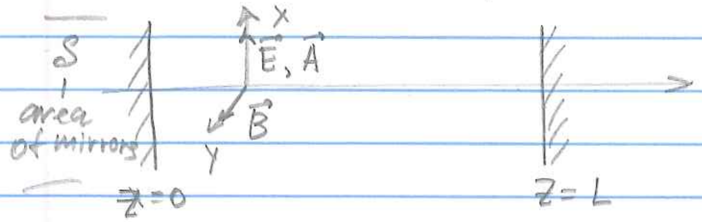
QED solution:

$$\hat{\vec{A}}(\vec{r},t) = \sum_k \left[\vec{A}_k(\vec{r},t)\,\hat{a}_k + \vec{A}_k^*(\vec{r},t)\,\hat{a}_k^+ \right]$$

where $\hat{a}_k$, $\hat{a}_k^+$ are lowering and raising operators for SHO

(usually called annihilation and creation operators in quantum optics)

Simplest case: plane EM wave b/w two perfect conductors



$\varepsilon = 1 \quad \mu = 1$

$\vec{E}(z=0) = \vec{E}(z=L) = 0$ boundary conditions

$$\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t} = 0 \quad \text{or} \quad \nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t} = 0$$

Spatial modes that automatically satisfy the boundary conditions: standing waves

$$\vec{A}, \vec{E} \propto \sin\left[\frac{\pi n z}{L}\right] \qquad n = 1, 2, \ldots$$

Single - mode approximation
(one particular standing wave is excited)
Let's pick $\vec{E} = (E_x, 0, 0)$, $\vec{B} = (0, B_y, 0)$, $\vec{A} = (A_x, 0, 0)$

$$E_x(\vec{r}, t) = E_x(z, t) = \mathcal{E}(t) \cdot \sin(kz) \qquad k = \frac{\pi n}{L} = \frac{\omega}{c}$$

At that point we are going to "guess" the normalization

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q(t) \sin kz$$

V – volume of the cavity

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \frac{\partial B_y}{\partial z} = \frac{1}{c^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q}(t) \sin kz$$

$$B_y = -\frac{1}{(ck)}\sqrt{\frac{2\omega^2}{V\varepsilon_0 c^2}}\, \dot{q}(t) \cos kz = -\sqrt{\frac{2}{V\varepsilon_0 c^2}}\, \dot{q}(t) \cos kz$$

($ck = \omega$)

When moving to quantum case
$q \to \hat{q}$, $\dot{q} \to \hat{p}$ (canonical position and momentum)

Hamiltonian

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2 \right] =$$

$$= \frac{1}{2}\varepsilon_0 \left(\frac{2\omega^2}{V\varepsilon_0}\right) q^2(t) \int dV \cdot \sin^2 kz + \frac{1}{2\mu_0}\frac{2}{V\varepsilon_0}(\dot{q}(t))^2 \int dV \cos^2 kz$$

$$\int_V \sin^2 kz \; dz\,dx\,dy = S\int_0^L \sin^2 kz\, dz = S\cdot\frac{L}{2} = \frac{V}{2}$$

$$H = \frac{\omega^2}{V} q^2(t)\cdot\frac{V}{2} + \frac{1}{V}(\dot{q})^2\frac{V}{2} = \frac{1}{2}\left((\dot{q})^2 + \omega^2 q^2\right)$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right) \qquad \text{SHO !}$$

$$[\hat{q}, \hat{p}] = i\hbar$$

Annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

$$\hat{E}_x = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(\hat{a} + \hat{a}^+)\sin kz$$

$$\hat{B}_y = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(\hat{a} - \hat{a}^+)\cos kz$$

$$[a, a^+] = 1 \longrightarrow aa^+ - a^+a = 1$$

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = i\omega\,[\hat{a}^+\hat{a}, \hat{a}] = i\omega\,(\hat{a}^+\hat{a}\hat{a} - \hat{a}\overbrace{\hat{a}^+\hat{a}}^{1+\hat{a}^+\hat{a}})$$

$$= -i\omega\hat{a}$$

$$\hat{a}(t) = e^{i\omega t}\hat{a}$$

Plane e-m linearly polarized wave inside the cavity

$$\begin{cases} \hat{E}_x = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) \sin kz \\ \hat{B}_y = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} - \hat{a}^+ e^{i\omega t}\right) \cos kz \end{cases}$$

Fock states (or number states)

$\hat{n} = \hat{a}^+\hat{a}$ $\qquad \hat{n}|n\rangle = n|n\rangle$

$|n\rangle$ - eigenstates of the number operator, describing the state with known number of photons

$|0\rangle$ - vacuum state (no photons)

$|1\rangle$ - single-photon state

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

$$E_n = n\hbar\omega + \frac{\hbar\omega}{2}$$

Why $\hat{a}$ is annihilation operator?

$$\hat{n}(\hat{a}|n\rangle) = \underbrace{\hat{a}^+\hat{a}}_{\hat{a}\hat{a}^+ - 1}(\hat{a}|n\rangle) = (\hat{a}\underbrace{\hat{a}^+\hat{a}}_{\hat{n}} - \hat{a})|n\rangle =$$

$$= \hat{a}\underbrace{(\hat{n}|n\rangle)}_{=n|n\rangle} - \hat{a}|n\rangle = (n-1)\,\hat{a}|n\rangle$$

[one photon is gone after $\hat{a}$]

$\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with eigen value of $(n-1)$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Correspondingly

$$\hat{n}\,(\hat{a}^+|n\rangle) = \hat{a}^+\hat{a}\,(\hat{a}^+|n\rangle) = \hat{a}^+(\hat{a}^+\hat{a}+1)|n\rangle =$$
$$= (n+1)\,\hat{a}^+|n\rangle$$

$\hat{a}^+|n\rangle$ is an eigenstate of $|n\rangle$ with eigenvalue $(n+1)$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

[one extra photon after $\hat{a}^+$]

Quadrature operators

Electric field

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\,(\hat{a}e^{-i\omega t} + \hat{a}^+e^{i\omega t})\sin kz$$

Such written $E_x(z,t)$ is a complex operator, so unphysical

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{a}(\cos\omega t - i\sin\omega t) + \hat{a}^+(\cos\omega t + i\sin\omega t)\right] \times \sin kz =$$

$$= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[(\hat{a}+\hat{a}^+)\cos\omega t - i(\hat{a}-\hat{a}^+)\sin\omega t\right] \times \sin kz =$$

$$= 2\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{X}_1\cos\omega t + \hat{X}_2\sin\omega t\right] \times \sin kz$$

$$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^+)$$
$$\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^+)$$

] Quadrature operators

They represent "real" and "imaginary" parts of e-m field, oscillating with frequency ω with $\pi/2$ phase lag between each other.

We can also make connections with canonical position and momentum operators for e-m field

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} + i\hat{p})$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} - i\hat{p})$$

Thus

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^+) = \frac{\omega}{\sqrt{2\hbar\omega}} \hat{q} \quad \text{(like electric field)}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^+) = \frac{1}{\sqrt{2\hbar\omega}} \hat{p} \quad \text{(like magnetic field)}$$

Thus, two quadratures are expected to behave as quantum position and momentum → not commute!

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$