Reminder:

Semi-classical light-atom interaction

$$\hat{H}_{int} = -\hat{\vec{d}} \cdot \vec{E}$$

$$\hat{\vec{d}} = -e \cdot \hat{\vec{r}} \quad \text{dipole moment – atomic operator}$$

$$\vec{E} = \hat{E}_0 \cos(kz - \omega t) \quad \text{– classical vector}$$

Fully quantum light-atom interaction

$$\hat{H}_{int} = -\hat{\vec{d}} \cdot \hat{\vec{E}}$$

$$\hat{\vec{E}} = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \vec{e} \, [\underbrace{\hat{a} e^{-i\omega t}}_{\hat{a}(t)} - \underbrace{\hat{a}^+ e^{i\omega t}}_{\hat{a}^+(t)}]$$

$$\hat{\vec{E}} = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \vec{e} \, [\hat{a}(t) - \hat{a}^+(t)]$$

$$\hat{H}_{int} = -i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \, (\hat{\vec{d}} \cdot \vec{e}) \, (\hat{a} - \hat{a}^+)$$

$$\hat{H}_{tot} = \underbrace{\hat{H}_a + \hat{H}_F}_{\hat{H}_0} + \hat{H}_{int}$$

$$\hat{H}_F = \hbar\omega \left(\hat{a}^+ \hat{a} + \frac{1}{2}\right)$$

Convenient way to express atomic operator, using its eigenstate

$$\hat{H}_a |i\rangle = E_i |i\rangle \quad \Rightarrow \quad \hat{H}_a = \sum_i E_i |i\rangle\langle i|$$

since $\sum_i |i\rangle\langle i| = \mathbb{1}$ complete set

also, since $\langle i | \hat{\vec{d}} \vec{e} | j \rangle = \rho_{ij}$

$$(\hat{\vec{d}} \vec{e}) = \sum_{ij} |i\rangle\langle i| \hat{\vec{d}} \hat{\vec{e}} | j \rangle\langle j| = \sum_{ij} \rho_{ij} |i\rangle\langle j|$$

Initial conditions $\underbrace{|a\rangle}_{\substack{\uparrow \\ \text{initial atomic state}}} \underbrace{|n\rangle}_{\substack{\nwarrow \\ \text{initial photonic state}}} = |\psi_i\rangle$

eigenstate of $\hat{H}_0$

$$\hat{H}_{int}|\psi_0\rangle = \left[-i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\right]\left(\sum_{ij} p_{ij}|i\rangle\langle j|\right)(\hat{a}-\hat{a}^{+})|a\rangle|n\rangle$$

$$= \left[-i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\right] \underbrace{\sum_i p_{ia}|i\rangle}_{\substack{|i\rangle \neq |a\rangle \\ \text{sinnce } p_{aa}=0}} (\underbrace{\sqrt{n}|n-1\rangle}_{\substack{\uparrow \\ \text{a photon absorbed}}} - \underbrace{\sqrt{n+1}|n+1\rangle}_{\substack{\uparrow \\ \text{a photon emitted}}})$$

Light atom interaction makes the atom dange its state by absorbing or emitting a photon

Notice that even if $n=0$ - vacuum state $|0\rangle$, there is still a non-zero probability to emit a photon (if there are allowed transitions $p_{ia} \neq 0$)
→ spontaneous emission

Let's now consider the possible state of the system if an atom starts at the state $|a\rangle$ and end up at the state $|b\rangle$ (i.e if only $p_{ab} \neq 0$)

$$|\psi(t)\rangle = c_a(t)\,|a\rangle|n\rangle\, e^{-iE_a t/\hbar}\, e^{-in\omega t} +$$

$$+\; c_b(t)\,|b\rangle|n-1\rangle\, e^{-iE_b t/\hbar}\, e^{-i(n-1)\omega t} +$$

$$+\; c_b'(t)\,|b\rangle|n+1\rangle\, e^{-iE_b t/\hbar}\, e^{-i(n+1)\omega t}$$

Perturbation theory

$$c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \langle b, n-1|\hat{H}_{int}|a,n\rangle\, e^{+i(\omega_{ba}-\omega)t'}\,dt'$$

$$c_b'^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \langle b, n+1|\hat{H}_{int}|a,n\rangle\, e^{i(\omega_{ba}+\omega)t'}\,dt'$$

where $\omega_{ba} = E_b - E_a$

$$\langle b, n\pm 1|\hat{H}_{int}|a,n\rangle = \left[-i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\right]\rho_{ba}\begin{cases}-\sqrt{n+1} & \text{"+"}\\ \sqrt{n} & \text{"-"}\end{cases}$$

$\mathcal{E}_0$ – electric field of a single-photon

$$g_{ba} = \frac{\mathcal{E}_0\,\rho_{ba}}{\hbar}$$ single-photon Rabi frequency (or coupling constant)

$$\langle b, n\pm 1|\hat{H}_{int}|a,n\rangle = -i\hbar\, g_{ba}\begin{cases}-\sqrt{n+1}\\ \sqrt{n}\end{cases}$$

Total probability amplitude to transition $|a\rangle \to |b\rangle$

$$c_b(t) + c_b'(t) = i g_{ba}\left\{\sqrt{n}\,\frac{e^{i(\omega-\omega_{ba})t}-1}{\omega-\omega_{ba}} - \sqrt{n+1}\,\frac{e^{i(\omega+\omega_{ba})t}-1}{\omega+\omega_{ba}}\right\}$$

Rotating wave approximation

if $E_b > E_a$ $\omega_{ba} > 0$

the first term dominates (photon is absorbed)

$$P_b \propto |g_{ba}|^2 \cdot n$$

if $E_b < E_a$ $\omega_{ba} < 0$

the second term dominates (photon is emitted)

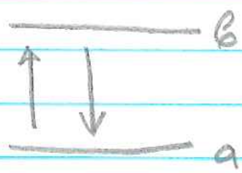
$$P_b \propto |g_{ba}|^2 (n+1)$$

$$\frac{P_{emission}}{P_{absorption}} = \frac{n+1}{n}$$

$n=0$ - spontaneous emission

$n>0$ - stimulated emission

We can finally get the black-body radiation distribution law from the first principle



$$\omega_{ba} = \omega$$

$$P_{abs} = n|g_{ba}|^2$$

$$P_{emis} = (n+1)|g_{ab}|^2$$

(but $g_{ba} = g_{ab}^*$)

at thermal equilibrium

b ——
$$\frac{dN_a}{dt} = -N_a P_{abs} + N_b P_{em} = 0$$

a ——
$$\frac{dN_b}{dt} = -N_b P_{em} + N_a P_{abs} = 0$$

$$N_a P_{abs} = N_b P_{em}$$

$$\frac{P_{em}}{P_{abs}} = \frac{n+1}{n} = \frac{N_a}{N_b} = e^{E_b - E_a/kT} = e^{\hbar\omega/kT}$$

$$1 + \frac{1}{n} = e^{\hbar\omega/kT}$$

$$n = \frac{1}{1 - e^{\hbar\omega/kT}}$$

Since typically the radiation will not be in a number state, but a combination of the number states $|\psi\rangle = \sum c_n |n\rangle$, and in the equation above $n \to \bar{n}$

$$\bar{n} = \frac{1}{1 - e^{\hbar\omega/kT}}$$