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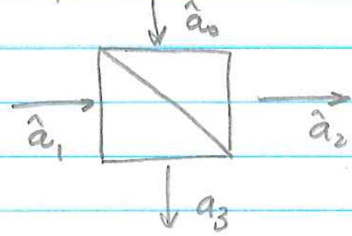
Interferometry with quantum states

Recall a "rules" for beam-splitter operation



50/50 Beam splitter

$$\begin{pmatrix}\hat{a}_2\\ \hat{a}_3\end{pmatrix} = \hat{U}^{+}_{BS}\begin{pmatrix}\hat{a}_0\\ \hat{a}_1\end{pmatrix}\hat{U}_{BS}$$

$$U_{BS} = e^{i\pi/4\,(\hat{a}_0^{+}\hat{a}_1 + \hat{a}_0\hat{a}_1^{+})}$$

$$\left.\begin{aligned}\hat{a}_2 &= \frac{1}{\sqrt{2}}(\hat{a}_0 + i\hat{a}_1)\\ \hat{a}_3 &= \frac{1}{\sqrt{2}}(i\hat{a}_0 + \hat{a}_1)\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned}\hat{a}_0 &= \frac{1}{\sqrt{2}}(\hat{a}_2 - i\hat{a}_3)\\ \hat{a}_1 &= \frac{1}{\sqrt{2}}(-i\hat{a}_2 + \hat{a}_3)\end{aligned}\right.$$

Another important element - phase shifter

$$U_\theta = e^{i\hat{a}^{+}\hat{a}\cdot\theta}$$

$$U_\theta|\alpha\rangle = e^{i\hat{a}^{+}\hat{a}\theta}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n e^{in\theta}}{\sqrt{n}}|n\rangle$$

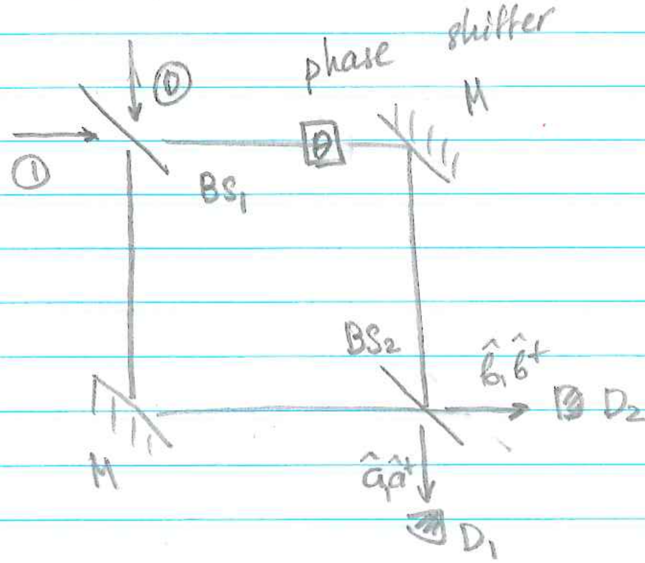
$$= \sum_{n=0}^{\infty}\frac{(\alpha e^{i\theta})^n}{\sqrt{n}}|n\rangle = |\alpha e^{i\theta}\rangle = |\alpha'\rangle$$

As expected from classical E&M, the phase shifter changes the phase of the EM field by θ.

However $U_\theta|n\rangle = \underline{\underline{e^{in\theta}}}|n\rangle$

n-times larger phase-shift

Simple Mach-Zender interferometer



Coherent state as an input.
Question: how accurately one can measure the phase θ?

Input $\psi_i = |0\rangle|\alpha\rangle$
After the first BS: $\rightarrow |\frac{i\alpha}{\sqrt{2}}\rangle|\frac{\alpha}{\sqrt{2}}\rangle$

Phase-shifter acts on one of the arms

$$\rightarrow |\frac{i\alpha}{\sqrt{2}}e^{i\theta}\rangle|\frac{\alpha}{\sqrt{2}}\rangle$$

Second beamsplitter

$$\psi_f \Rightarrow \left|\frac{i(e^{i\theta}+1)\alpha}{\sqrt{2}}\right\rangle\left|\frac{(e^{i\theta}-1)\alpha}{\sqrt{2}}\right\rangle$$

Signal detected on the photodetectors proportional to the mean values of photons. Measured difference

$$\hat{O} = \hat{a}^+\hat{a} - \hat{b}^+\hat{b}$$

$$\langle\psi_f|\hat{O}|\psi_f\rangle = \frac{|(e^{i\theta}+1)|^2}{2}|\alpha|^2 + \frac{|e^{i\theta}-1|^2}{2}|\alpha|^2 =$$

$= |\alpha|^2\cos\theta$ same as classical

There is always an uncertainty

$$\Delta O = \sqrt{\langle O^2\rangle - \langle\hat{O}\rangle^2}$$

For a coherent state $\Delta n = \bar{n}^{1/2} = |\alpha|$

Uncertainty of the phase measurements

$$\Delta\theta = \frac{\Delta O}{\left(\frac{\partial O}{\partial\theta}\right)} = \frac{1}{|\alpha||\sin\theta|} = \frac{1}{\sqrt{\bar{n}}|\sin\theta|}$$

Max sensitivity $|\sin\theta| \sim 1$

$\Delta\theta_{min} = \frac{1}{\sqrt{\bar{n}}}$ ← shot noise level

This is the fundamental limit for measurements with <u>coherent</u> input state

Fundamental limit : Heisenberg uncertainty

$\Delta\theta \sim \frac{1}{\bar{n}}$ ← Holy grail of quantum measurements

Single photon interferometer

Input state $|0\rangle|1\rangle$

After the first BS: $\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + i|1\rangle|0\rangle)$

After the phase shifter $\frac{1}{\sqrt{2}}(e^{i\theta}|0\rangle|1\rangle + i|1\rangle|0\rangle)$

After the second BS:

$$\left[\begin{array}{l} |0\rangle|1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + i|1\rangle|0\rangle) \\ |1\rangle|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1\rangle|0\rangle + i|0\rangle|1\rangle) \end{array}\right]$$ each component transforms

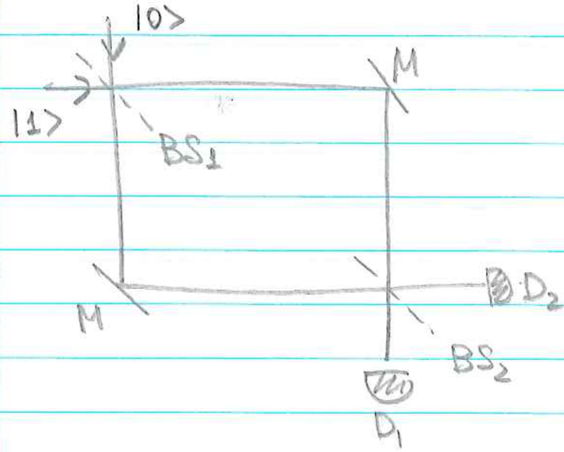
$$\psi_{fin} = \frac{1}{2}\left\{ e^{i\theta}(|0\rangle|1\rangle + i|1\rangle|0\rangle) + i(|1\rangle|0\rangle + i|0\rangle|1\rangle)\right\} =$$

$$= \frac{1}{2}\left\{ (e^{i\theta}-1)\,|0\rangle|1\rangle + i(e^{i\theta}+1)\,|1\rangle|0\rangle \right\}$$

Each detector will click with probabilities $P_{1,2} = \frac{1}{2}(1 \pm \cos\theta)$

$\langle \psi_{fin}|\hat{O}|\psi_{fin}\rangle = \cos\theta$ (similar to coherent case)

Interaction-free measurements



θ is set to zero

After BS 1

$$\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + i|1\rangle|0\rangle\right)$$

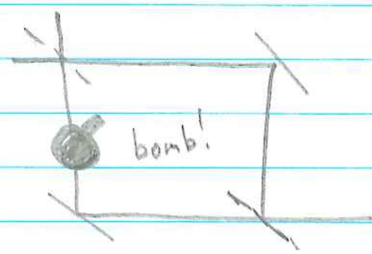
After BS 2

$$\frac{1}{2}\left\{\left(\cancel{|0\rangle|1\rangle} + i|1\rangle|0\rangle\right) + i\left(|1\rangle|0\rangle + \cancel{i|0\rangle|1\rangle}\right)\right\}$$

$$= |1\rangle_{1}|0\rangle_{2}$$

Only D_1 detects photons, and never D_2.

Now we put something in one of the arms



After BS 1

$$\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + i|1\rangle|0\rangle\right)$$

the Bomb scatters all the photons, so after it (and before the BS2)

$$\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + i|1\rangle|0\rangle\right)$$

After the second BS

$$\Psi_{fin} = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{i}{2}\left(|1\rangle|0\rangle + i|0\rangle|1\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{i}{2}|1\rangle|0\rangle - \frac{1}{2}|0\rangle|1\rangle$$

Possible outcomes: nothing - 50% (but bomb is set off!)
D1 clicks - 25% (no information)
D2 clicks - 25% (bomb is there!)
even though the photon took the other path