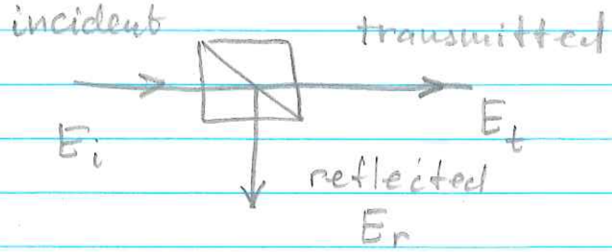
Quantum mechanics of a beam splitter

Beam splitter is a device to divide (or combine) optical fields

Classical beam splitter

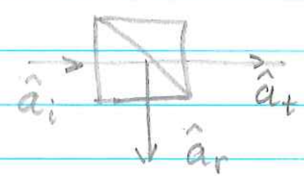


$E_t = tE_i$ $\qquad |t|^2+|r|^2=1$

$E_r = rE_i$ $\qquad$ energy conservation

Quantum $\quad E \to \hat{a}$

assumption



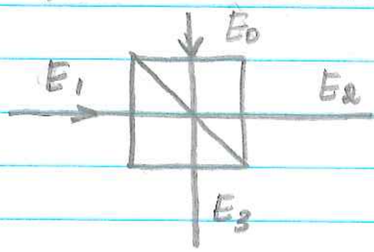
$\hat{a}_t = t\hat{a}_i$

$\hat{a}_r = r\hat{a}_i$ $\quad$?

$$[\hat{a}_t, \hat{a}_t^+] = [t\hat{a}_i, t^*\hat{a}_i^+] = |t|^2 [\hat{a}_i, \hat{a}_i^+] = |t|^2 \leq 1 \quad \underline{???}$$

Our assumption is not correct → we forgot the fourth port!
In classical treatment vacuum is neglectable. In quantum treatment vacuum is an active participant!

More accurate picture



	E_1	E_0
E_2	t	r'
E_3	r	t'

differences b/w (r,t) and (r',t') reflect possible phase differences

$$E_2 = tE_1 + r'E_0$$
$$E_3 = rE_1 + t'E_0 \quad \Rightarrow \quad \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}$$

Assuming now $E_i \rightarrow \hat{a}_i$

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}$$

Checking the commutation relationships

$$[\hat{a}_2, \hat{a}_2^+] = [t'\hat{a}_0 + r\hat{a}_1, t'^*\hat{a}_0^+ + r^*\hat{a}_1^+] =$$

$$= |t'|^2 [\hat{a}_0, \hat{a}_0^+] + |r|^2 [\hat{a}_1, \hat{a}_1^+] = |t'|^2 + |r|^2 = 1$$

If $|t'| = |t|$, $|r'| = |r|$

$$[\hat{a}_2, \hat{a}_3^+] = [t'\hat{a}_0 + r\hat{a}_1, r'^*\hat{a}_0 + t^*\hat{a}_1] =$$

must be

$$= t'r'^* [\hat{a}_0, \hat{a}_0^+] + rt^* [\hat{a}_1, \hat{a}_1^+] = t'r'^* + rt^* = 0$$

A simple 50/50 beamsplitter (one dielectric layer)

$$\hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{a}_0 + i\hat{a}_1) \qquad \hat{a}_3 = \frac{1}{\sqrt{2}}(i\hat{a}_0 + \hat{a}_1)$$

Write as a unitary transformation

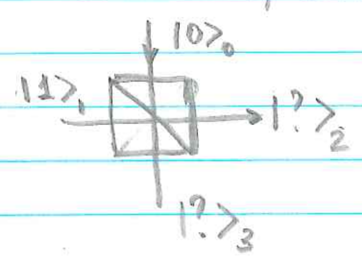
$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \hat{U}^+ \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \hat{U}$$

$$\hat{U} = e^{i\pi/4(\hat{a}_0^+ \hat{a}_1 + \hat{a}_0 \hat{a}_1)}$$

Heisenberg representation:
the state of the incident beam is invariant, the mode operators are modified

Schrödinger representation: operators are invariant, but states are changed

Example1: a single photon on a beam splitter



Handy thought:
vacuum is vacuum
$|0\rangle_0 |0\rangle_1 \xrightarrow[BS]{} |0\rangle_2 |0\rangle_3$

Initial state $|\psi\rangle = \hat{a}_1^+ |0\rangle_0 |0_1\rangle$

BS action

$$\downarrow \qquad \downarrow$$

$$\frac{1}{\sqrt{2}}(i\hat{a}_2^+ + \hat{a}_3^+)|0\rangle_2|0\rangle_3$$

Final state $|\psi'\rangle = \frac{1}{\sqrt{2}}(i|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle)$

A photon goes one way or another with 50% probability

Example 2: coherent state $|\alpha\rangle$

Initial state: $|\psi\rangle = |0\rangle_0 |\alpha\rangle_1 = \hat{D}_1(\alpha)|0\rangle_0 |0\rangle_1$

$$\hat{D}_1(\alpha) = e^{\alpha \hat{a}_1^+ - \alpha^* \hat{a}_1}$$

BS action

$$e^{\frac{\alpha}{\sqrt{2}}(i\hat{a}_2^+ + \hat{a}_3^+) - \frac{\alpha^*}{\sqrt{2}}(-i\hat{a}_2 + \hat{a}_3)} =$$

$$= \underbrace{e^{\frac{i\alpha}{\sqrt{2}}\hat{a}_2^+ - \left(\frac{i\alpha}{\sqrt{2}}\right)^* \hat{a}_2}}_{\hat{D}_2\left(\frac{i\alpha}{\sqrt{2}}\right)} \cdot \underbrace{e^{\frac{\alpha}{\sqrt{2}}\hat{a}_3^+ - \frac{\alpha}{\sqrt{2}}\hat{a}_3}}_{\hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)}$$

Output:

$$|\psi'\rangle = \hat{D}_2\left(\frac{i\alpha}{\sqrt{2}}\right)\hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)|0\rangle_2 |0_3\rangle = \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3$$

Equally split two coherent states with a relative phase shift

Example 3: Test for single-photons indistinguishibility (Hong-Ou-Mandel test)

Initial state $|1\rangle_0 |1\rangle_1 = \hat{a}_0^+ \hat{a}_1^+ |0\rangle_0 |0\rangle_1$

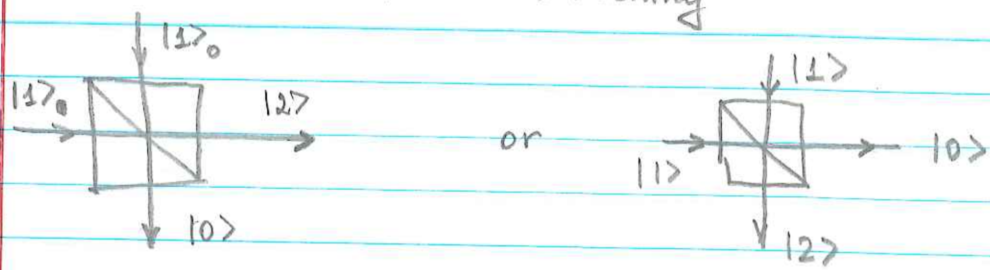
$$\hat{a}_0^+ \hat{a}_1^+ = \frac{1}{2}(\hat{a}_2^+ + i\hat{a}_3^+)(i\hat{a}_2^+ + \hat{a}_3^+) = \frac{i}{2}\left[(\hat{a}_2^+)^2 + (\hat{a}_3^+)^2\right] +$$

$$\cancel{+\frac{1}{2}\hat{a}_2^+ \hat{a}_3^+ - \frac{1}{2}\hat{a}_2^+ \hat{a}_3^+} = \frac{i}{2}\left[(\hat{a}_2^+)^2 + (\hat{a}_3^+)^2\right]$$

Final state $\frac{i}{2}\left[(\hat{a}_2^+)^2 + (\hat{a}_3^+)^2\right]|0\rangle_2 |0_3\rangle =$

$$= \frac{i}{\sqrt{2}}\left(|2\rangle_2 |0\rangle_3 + |0\rangle_2 |2\rangle_3\right)$$

Photon bunching



The probability to detect photons at both output simultaneously is zero, (if, of coarse, the initial photons are indistinguishable)