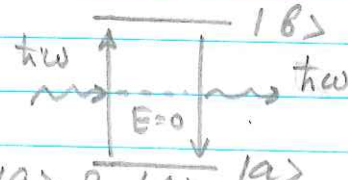
Quantum light - atom interaction in a two-level system

$$\hat H = \underbrace{\sum_i E_i |i\rangle\langle i|}_{\text{atom only}} + \underbrace{\hbar\omega\left(\hat a^+\hat a + \frac{1}{2}\right)}_{\text{photons only}} + \underbrace{\sum_{ij}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, p_{ij}\, |i\rangle\langle j| \times (\hat a + \hat a^+)}_{\text{interaction}}$$

Two-level system



$\hat H_a = E_a |a\rangle\langle a| + E_b |b\rangle\langle b|$

let's pick $E=0$ b/w $|a\rangle$ & $|b\rangle$

$$\hat H_a = -\frac{1}{2}(E_b - E_a)|a\rangle\langle a| + \frac{1}{2}(E_b - E_a)|b\rangle\langle b| =$$

$$= \frac{1}{2}\hbar\omega_0 \underbrace{\left(|b\rangle\langle b| - |a\rangle\langle a|\right)}_{\hat\sigma_3 \text{ – inversion operator}} = \frac{1}{2}\hbar\omega_0 \sigma_3$$

$$\hat H_i = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(p_{ab}|a\rangle\langle b| + p_{ba}|b\rangle\langle a|\right)(\hat a + \hat a^+) =$$

assuming p_{ab} is real

$$= \hbar g\,(\underbrace{|a\rangle\langle b|}_{\hat\sigma_-} + \underbrace{|b\rangle\langle a|}_{\hat\sigma_+})(\hat a + \hat a^+)$$

atomic transition operator

Three atomic operators $\hat\sigma_\pm$ and $\hat\sigma_3$ obey the Pauli matrix commutations

$$[\hat\sigma_+, \hat\sigma_-] = \hat\sigma_3$$
$$[\hat\sigma_3, \hat\sigma_\pm] = 2\hat\sigma_\pm$$

$$\hat H_i = -i\hbar g\,(\hat\sigma_+ + \hat\sigma_-)(\hat a - \hat a^+)$$

$\langle\psi|\sigma_+|\psi\rangle = \langle\psi|b\rangle\langle a|\psi\rangle = c_b^* c_a = \rho_{ab}$

$\langle\psi|\sigma_3|\psi\rangle = \langle\psi|b\rangle\langle b|\psi\rangle - \langle\psi|a\rangle\langle a|\psi\rangle = |b|^2 - |a|^2 = \rho_{bb} - \rho_{ab}$

For a two-level system

$$\hat{H} = \frac{1}{2}\hbar\omega_0 \hat{\sigma}_3 + \hbar\omega\, \hat{a}^+\hat{a} + \hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}^+ + \hat{a})$$

as we discussed last time if $\omega \approx \omega_0$, the two plausible scenarios: photon is aborbed and an atom goes from $|g\rangle \to |e\rangle$, and reversed: photon is emitted and an atom goes from $|e\rangle \to |g\rangle$

$\longrightarrow$ $\cancel{\hat{\sigma}_+\hat{a}^+}$ and $\cancel{\hat{\sigma}_-\hat{a}}$

RWA Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0 \hat{\sigma}_3 + \hbar\omega\, \hat{a}^+\hat{a} + \hbar g(\hat{\sigma}_-\hat{a}^+ + \hat{\sigma}_+\hat{a})$$

no interaction: eigenstates are $\hat{H}|a,n\rangle = -\frac{1}{2}\hbar\omega_0 + \hbar\omega$
$\hat{H}|b,n\rangle = \frac{1}{2}\hbar\omega_0 + \hbar\omega$

We will assume a <u>closed</u> system:

- an atom can only be found in the states $|a\rangle$ & $|b\rangle$ $\quad |a\rangle\langle a| + |b\rangle\langle b| = \mathbb{1}$
- energy is conserved, so the total number of excitations is constant $\hat{N}_e = |b\rangle\langle b| + \hat{a}^+\hat{a}$

Two coupled states

$|b,n\rangle \quad |1\rangle \quad E^{(0)}_{1n} = \frac{1}{2}\hbar\omega_0 + \hbar\omega \cdot n = \hbar\omega(n+\frac{1}{2}) + \frac{1}{2}\hbar(\omega_0 - \omega)$

$|a,n+1\rangle \quad |2\rangle \quad E^{(0)}_{2n} = -\frac{1}{2}\hbar\omega_0 + \hbar\omega(n+1) = \hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\hbar(\omega_0 - \omega)$

$\Delta = \omega_0 - \omega$

We can thus decompose the Hamiltonian to the pairs of coupled states for each photon number state

$$\hat{H} = \sum_n \hat{H}_n$$

$$\hat{H}_n = \hbar\omega(n+\tfrac{1}{2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \tfrac{1}{2}\hbar\Delta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} +$$

$$+\hbar g\begin{pmatrix}0 & \sqrt{n+1}\\ \sqrt{n+1} & 0\end{pmatrix} = \underbrace{\hbar\omega(n+\tfrac{1}{2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\text{overall energy same for both states}} +$$

$$+\tfrac{1}{2}\hbar\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix}$$

Eigenstates of the hamiltonian

$$E_{2n} = \hbar\omega(n+1/2) - \tfrac{1}{2}\hbar\tilde{g}_n \qquad \tilde{g}_n = \sqrt{\Delta^2 + 4g^2(n+1)}$$

$$E_{1n} = \hbar\omega(n+1/2) + \tfrac{1}{2}\hbar\tilde{g}_n$$

generalized vacuum Rabi freq

Dressed states

$$|2n\rangle = \cos\theta_n\,|a,n\rangle - \sin\theta_n\,|b,n+1\rangle$$
$$|1n\rangle = \sin\theta_n\,|a,n\rangle + \cos\theta_n\,|b,n+1\rangle$$

$$\cos\theta_n = \frac{\tilde{g}_n - \Delta}{\sqrt{(\tilde{g}_n-\Delta)^2 + 4g^2(n+1)}} \qquad \sin\theta_n = \frac{2g\sqrt{n+1}}{\sqrt{(\tilde{g}_n-\Delta)^2 + 4g^2(n+1)}}$$

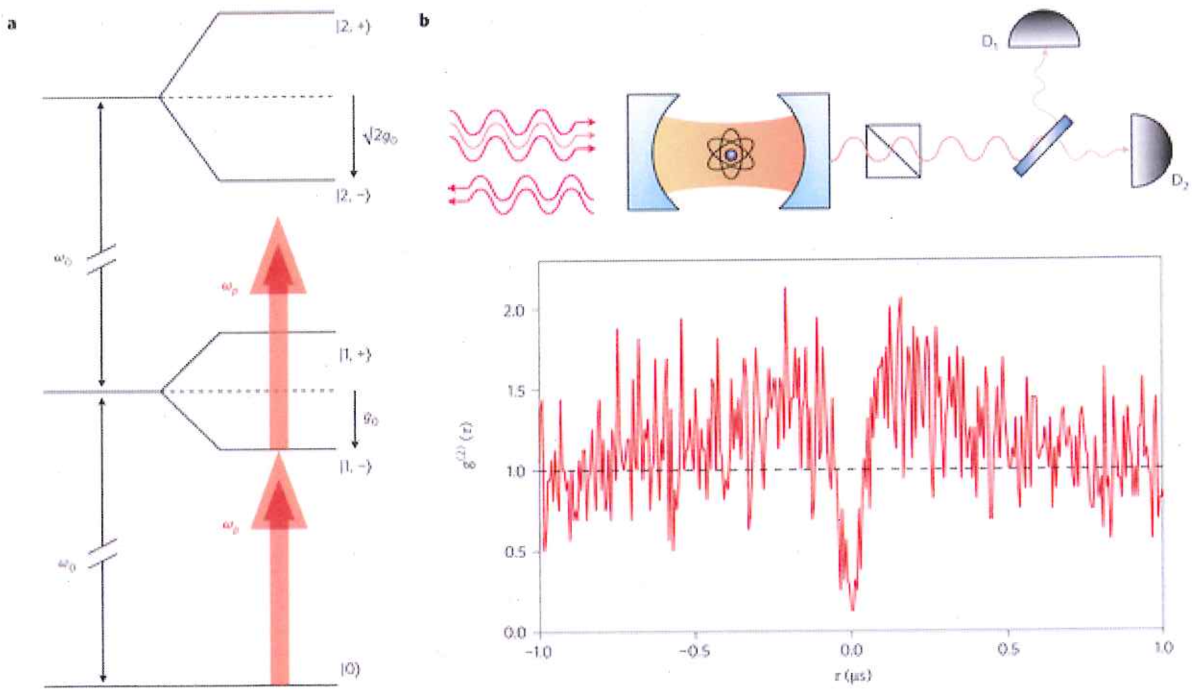
For $\Delta = 0$ $(\omega = \omega_0)$ $\quad E_{2n} = \hbar\omega(n+\tfrac{1}{2}) - \hbar g\sqrt{n+1}$

$$E_{1n} = \hbar\omega(n+\tfrac{1}{2}) + \hbar g\sqrt{n+1}$$

$$E_{1n} - E_{2n} = 2\hbar g\sqrt{n+1}$$

energy splitting of dressed states (even though the light is resonant with bare atomic states)

$$|2n\rangle = \tfrac{1}{\sqrt{2}}\left(|a,n\rangle - |b,n+1\rangle\right) \qquad |1n\rangle = \tfrac{1}{\sqrt{2}}\left(|a,n\rangle + |b,n+1\rangle\right)$$



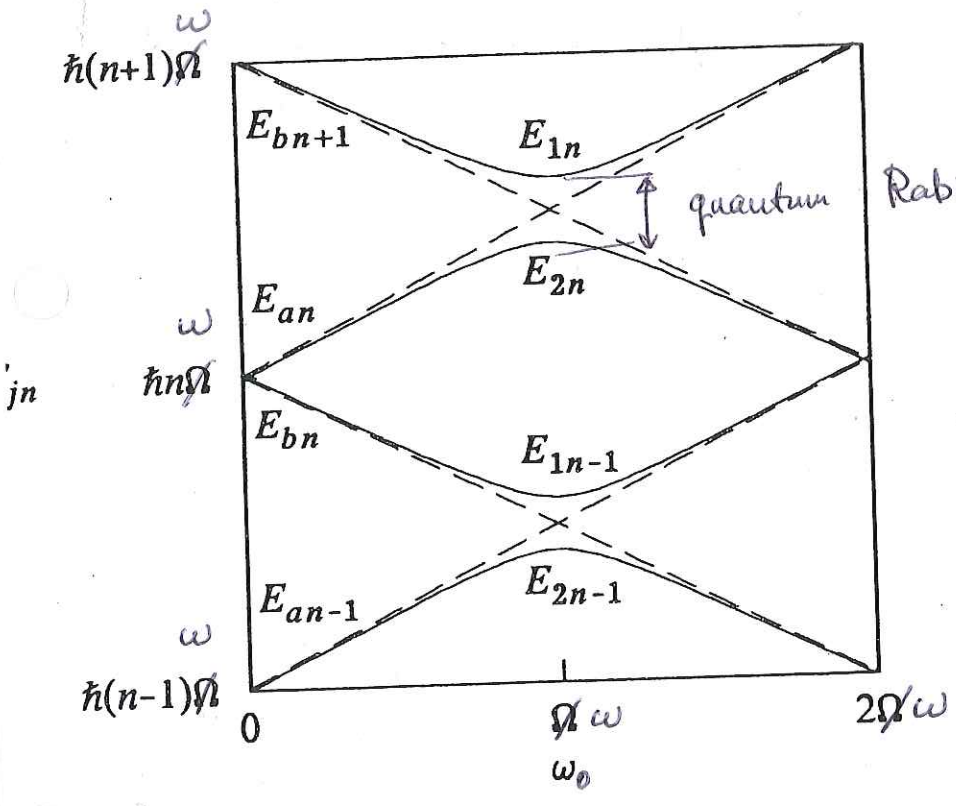


Fig. 14.1. Dressed atom energy level diagram. The dashed lines are the energy eigenvalues (14.11) for the atom-field system with no interaction energy. The solid lines include the atom-field interaction as in (14.14)