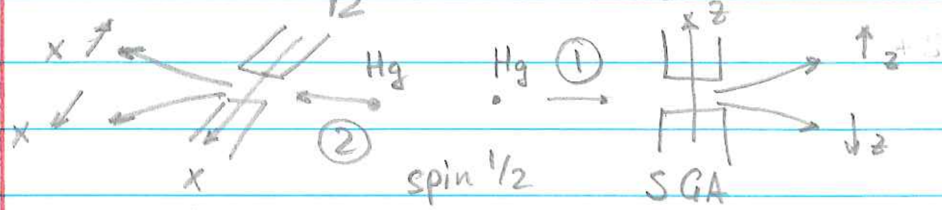
EPR paradox

Two-particle entangled state

$$|\psi^{(e)}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_1 \downarrow_2\rangle - |\downarrow_1 \uparrow_2\rangle\right) \qquad [|\psi^-\rangle \text{ Bell state}]$$



After a measurement

If $|\uparrow\rangle_z$ measured for 1 $\rightarrow$ $|\downarrow\rangle_z$ for 2

However, one can rewrite the state $|\psi^{(e)}\rangle$ in $|\pm\rangle_x$ basis

$$|\psi^{(e)}\rangle = \frac{1}{\sqrt{2}}\left(|+_1 -_2\rangle - |-\rangle_1 |+\rangle_2\right)$$

If $|+\rangle_x$ measured for 1 $\rightarrow$ $|-\rangle_x$ for 2

Thus, the state of 2 seem to change depending on 1 measurements, even with no interactions between them — non-locality. Wave-function formalism is inadequate since it seems that we cannot consistently describe the state of the system before measurements

Density matrix for particle 2

1 is measured in z-basis

$$\rho_2 = \langle\uparrow_1|\psi^{(e)}\rangle\langle\psi^{(e)}|\uparrow\rangle + \langle\downarrow_1|\psi^{(e)}\rangle\langle\psi^{(e)}|\downarrow\rangle =$$

$$= \frac{1}{2}\left(|\downarrow_2\rangle\langle\downarrow_2| + |\uparrow_2\rangle\langle\uparrow_2|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if ① is measured in x-basis

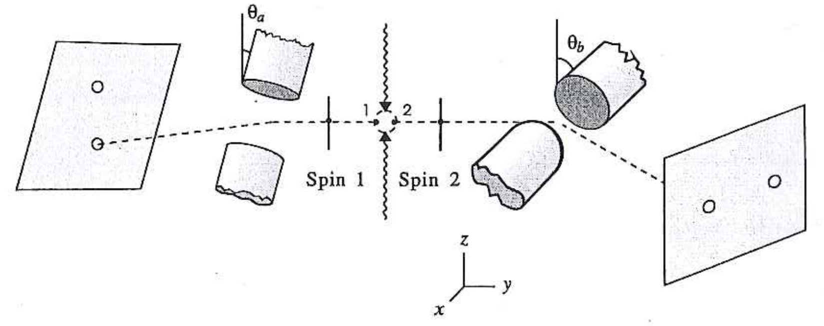
$$\rho_2 = \langle +_1 | \psi^{(2)} \rangle \langle \psi^{(2)} | +_1 \rangle + \langle -_1 | \psi^{(2)} \rangle \langle \psi^{(2)} | -_1 \rangle =$$
$$= \frac{1}{2} \left(|-_2\rangle\langle -_2| + |+_2\rangle\langle +_2| \right) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

same state for the particle ②

Entanglement raises the question of locality of the quantum mechanics — "spooky action at the distance"

Hidden variable theory → there is a parameter in each particle that controls the outcome of each measurement. Since we don't know its parameters, the outcome will see random to us, but is actually pre-set before particles part

Let's assume that the spin direction is pre-set, so any measurement results can be predicted beforehand.



We'll consider three possible measurement orientations: $\theta_a, \theta_b, \theta_c$; "+" - pass, "−" - block

Joint probability P_{ab}: ① passes through θ_a oriented detector, ② passes through θ_b oriented detector

$$P_{ab} = (\underbrace{\overset{a}{+}\;\overset{b}{-}\;\overset{c}{\bigcirc}}_{\text{particle 1}} \mid \underbrace{\overset{a}{-}\;\overset{b}{+}\;\overset{c}{\bigcirc}}_{\text{particle 2}})$$

anti-correlated

Analogously, $P_{bc} = (\bigcirc + - \mid \bigcirc - +)$

$P_{ac} = (+ \bigcirc - \mid - \bigcirc +)$

Logically

$P_{ab} = (+ - + \mid - + -) + (+ - - \mid - + +)$

$P_{bc} = (+ + - \mid - - +) + (- + - \mid + - +)$

$P_{ac} = (+ + - \mid - - +) + (+ - - \mid - + +)$

$$P_{ab} + P_{bc} = \overbrace{(+ + - \mid - - +)} + (- + - \mid + - +) + (+ - + \mid - + -) + \overbrace{(+ - - \mid - + +)}$$

$$= P_{ac} + (- + - \mid + - +) + (+ - + \mid - + -) \geq P_{ac}$$

$P_{ab} + P_{bc} \geq P_{ac}$ Bell's theorem!

Quantum calculation of the correlations in Bell's theorem

Spin rotation $|\theta\rangle = e^{-i\theta\hat{\delta}_y}|\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$

Projection operator $\hat{\pi}_\theta = |\theta\rangle\langle\theta|$

such that $P_\psi(\theta) = \langle\psi|\theta\rangle\langle\theta|\psi\rangle$ – probability of a particle at the state $|\psi\rangle$ do pass the detector

$$\hat{\pi}_\theta = \left(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle\right)\left(\cos\frac{\theta}{2}\langle\uparrow| + \sin\frac{\theta}{2}\langle\downarrow|\right) =$$

$$= \left[\cos^2\frac{\theta}{2}|\uparrow\rangle\langle\uparrow| + \sin^2\frac{\theta}{2}|\downarrow\rangle\langle\downarrow| + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\right]$$

$$= \frac{1}{2}\left(1 + \hat{\delta}_z\cos\theta + \hat{\delta}_x\sin\theta\right)$$

$$P_{ab} = \langle\psi^{(e)}|\hat{\pi}^{(1)}_{\theta_a}\hat{\pi}^{(2)}_{\theta_b}|\psi^{(e)}\rangle = \frac{1}{4}\left[1 - \cos(\theta_a - \theta_b)\right] =$$

$$= \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

Bell's inequality

$$\frac{1}{2}\sin^2\frac{(\theta_a - \theta_b)}{2} + \frac{1}{2}\sin^2\frac{(\theta_b - \theta_c)}{2} \geq \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_c}{2}\right)$$

For $\theta_a = 0$, $\theta_b = \pi/4$, $\theta_c = \pi/2$

we must compare LHS: $\sin^2 \pi/8 = \frac{2-\sqrt{2}}{4} \approx 0.15$

RHS: $\frac{1}{2}\sin^2\pi/4 = \frac{1}{4} = 0.25$

clearly Bell's inequality is violated!