**PHYS 481/690**

***Problem set # 4*** *(due November 2)*

**A1** *(10 points)*
In class we derived expression for Rabi oscillations assuming no decay of the atomic states. While usually it is impossible to properly include spontaneous emission is a wave-function description of the semi-classical light-atom interactions, in the case of the two-level atom the finite lifetime of the atomic levels can be described by adding phenomenological decay terms to the probability amplitude equations:

$$\begin{aligned}\dot{c}_e &= -\frac{\gamma}{2}c_e - i\frac{\Omega}{2}c_g \\ \dot{c}_g &= -\frac{\gamma}{2}c_g - i\frac{\Omega}{2}c_e\end{aligned} \tag{1}$$

where $\gamma$ is the decay rate. For an atom initially in the state $|e\rangle$, show that the inversion (differences in populations between the excited and the ground states) is $P_e - P_g = e^{-\gamma t}cos(\Omega t)$.

**A2** *(35 points)*
To include the finite lifetime of some of the atomic states properly, one has to use the density matrix formalism. For a pure state the density matrix is defines as $\hat{\rho} = |\psi\rangle\langle\psi|$.

*(a)* For a two-level atom the wave function is $|\psi\rangle = c_e|e\rangle + c_g|g\rangle$. Write down the expression for the four density matrix elements in terms of the wave function coefficients, and explain their physical meaning.

*(b)* Using the equations we derived in class for a two-level atom without decays

$$\begin{aligned}\dot{c}_e &= -i\frac{\Omega}{2}c_g \\ \dot{c}_g &= -i\frac{\Omega}{2}c_e,\end{aligned} \tag{2}$$

write down the differential equations for $\rho_{ee}, \rho_{gg}$ and $\rho_{eg}$. Solve them and verify that the solution is identical to the one we obtained for the wave functions.

*(c)* We now can properly include the decay rates for the atomic levels. The density matrix element in this case will be governed by the following equations:

$$\begin{aligned}\dot{\rho}_{ee} &= -\gamma\rho_{ee} - i(\Omega\rho_{ge} - c.c), \\ \dot{\rho}_{gg} &= -\gamma\rho_{gg} + i(\Omega\rho_{ge} - c.c), \\ \dot{\rho}_{eg} &= -\gamma\rho_{eg} + i\Omega(\rho_{ee} - \rho_{gg}),\end{aligned} \tag{3}$$

Calculate the inversion between the levels and compare your answer with the result of A1, assuming the atom is initially in the excited state.

*(d)* We can now also consider the situation with state $|e\rangle$ decaying into state $|g\rangle$:

$$\begin{aligned}\dot{\rho}_{ee} &= -\gamma\rho_{ee} - i(\Omega\rho_{ge} - c.c), \\ \dot{\rho}_{gg} &= \gamma\rho_{ee} + i(\Omega\rho_{ge} - c.c), \\ \dot{\rho}_{eg} &= -\gamma/2\rho_{eg} + i\Omega(\rho_{ee} - \rho_{gg}),\end{aligned} \tag{4}$$

Assuming the atom initially in the excited state, calculate the population inversion.

**A3** *(10 points)*
Using explicit expressions for the wave functions for the hydrogen atom, explain which transitions are possible between $1S$ state and various $m$-sublevels of $2P$ states, in case of $x$, $y$ and $\sigma_\pm$ polarized electromagnetic field. Assume that the light propagates along $z$ axis, and it is also the quantization direction for the atoms.