Exclusive $\pi^+\pi^-$ Electroproduction
HERMES/NIKHEF Group Meeting
11 December 2003

Keith Griffioen
NIKHEF and College of William and Mary

December 12, 2003

Abstract

I have reproduced Riccardo Fabbri’s analysis of exclusive $\pi^+\pi^-$ production in HERMES, and have written a toy Monte Carlo simulation to study the effect of acceptance on the determination of Legendre moments.
 Exclusive 2-Pion Production

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\[
M = \sum_{ij} \int dz \int dx_1 f^{T'}_{i/T}(x_1, x_1-x_B, t) H_{ij} \left( \frac{x_1}{x_B}, Q^2, z \right) \Phi^F_j(z)
\]

\[\Theta(1/Q)\] corrections

Factorization Theorem

Collins, Frankfurt, Strikman
PRD56 (97)2982.

no proven factorization Thm for \( \gamma^* \) (transverse photons) but amplitudes are down by \( 1/Q \) w.r.t. \( \gamma^* \)
\[ \rho \text{-like} \]
\[ \rho (770) \quad I^G(J^{PC}) = l^+ (1^{--}) \]
\[ f \text{-like} \]
\[ f_0 (980) \quad 0^+ (0^{++}) \]
\[ f_2 (1270) \quad 0^+ (2^{++}) \]

**Charge Conjugation**
\[ U_c \left[ A = 0 \right] = \eta_c \left[ A = 0 \right] \]
\[ \eta_c = \pm 1 \]

**Photons:**
\[ e^+ \nu_e \to -e^- \bar{\nu}_e \quad (\text{change reverses}) \]
\[ \nabla^2 A_\mu = \frac{e^m}{\mu} \rightarrow A_\mu \rightarrow -A_\mu \]
\[ \eta_c = -1 \text{ for } \gamma \]

**\( \overline{B} \): total w.f. antisymmetric**

reverse particles:
\[ \{ \text{spatial exchange} \quad (-1)^l \}
\[ \text{spin exchange} \quad (-1)^{l+3} \}
\[ \text{change exchange} \]

\[ \eta_c = (-1)^{l+3} \]

**\( \pi^+ \pi^- \): symmetric w.f. \Rightarrow \eta_c (1)^l = 1 \text{ (no spin)}**

**Isospin:**
\[ |I=1, I_3=1\rangle = \frac{\sqrt{2}}{2} \left( \pi^+ \pi^- - \pi^- \pi^+ \right) \]
\[ |I=0, I_3=0\rangle = \frac{1}{\sqrt{3}} \left( \pi^+ \pi^- - \pi^- \pi^+ \right) \]
\[ |I=1, I_3=-1\rangle = \frac{1}{\sqrt{2}} \left( \pi^0 \pi^- - \pi^- \pi^0 \right) \text{ (antisym.)} \]

odd \( l \) \Rightarrow \( I = 1 \)

even \( l \) \Rightarrow \( I = 0 \)
Exclusive $\pi\pi$

\[
\begin{align*}
T^\pi\pi &= \mathcal{T}^{I=0} + \mathcal{T}^{I=1} \\
T^{\pi^+\pi^-} &= \mathcal{T}^{I=0} \\
T^{\pi^0\pi^0} &= \mathcal{T}^{I=0} \\
T^{I=0} &\propto \mathcal{E}^{I=0} \{\text{integrated GPDs}\} \\
T^{I=1} &\propto \mathcal{E}^{I=1} \{\text{integrated GPDs}\} \\
\Phi^{I=0} &\propto f_0(m_{\pi\pi}) P_0(\cos \theta) + c f_2(m_{\pi\pi}) P_2(\cos \theta) \\
\Phi^{I=1} &\propto F_{\pi}(m_{\pi\pi}) P_1(\cos \theta) \\
\langle P_2 \rangle &= \frac{\int \sigma_2 P_2 \, dx}{\int \sigma \, dx} \\
\langle P_1 \rangle &= \frac{\frac{2}{3} b_{01} + \frac{4}{15} b_{12}}{b_{00} + b_{11} + b_{22}} \\
\langle P_3 \rangle &= \frac{\frac{6}{35} b_{12}}{b_{00} + b_{11} + b_{22}} \\
\langle P_1 \rangle - \frac{4}{9} \langle P_3 \rangle &= \frac{\frac{2}{3} b_{01}}{b_{00} + b_{11} + b_{22}} \\
f_\pi(m_{\pi\pi}) &= \exp \int i \mathcal{S}_0^\pi(m_{\pi\pi}) + \frac{m_{\pi\pi}^2}{\pi} \left( -1 \right) \int_{4m_{\pi}^2}^{\infty} ds \frac{\mathcal{S}_\pi^\pi(s)}{s(s-m_{\pi\pi}^2-i\epsilon)} \\
\pi\pi\pi &\text{ phase shifts}
\end{align*}
\]
Formalism is only for $\gamma_L^*$ in 1st order

In general $\sigma = \sum_{J J', L L'} p_{J J'} Y_{J J} (\theta, \phi) Y_{J' L'} (\theta, \phi)$

Re-express as $\sigma = \sum_{L M} q_{LM} Y_{LM} (\theta, \phi)$

$$a_{10} = \frac{1}{14\pi} \left\{ 4 \sqrt{2} p_{11}^{21} + 4\sqrt{5} p_{00}^{21} + 2 p_{00}^{10} \right\}$$

$$a_{30} = \frac{1}{14\pi} \left\{ -12 \sqrt{135} p_{11}^{21} + 6 \sqrt{35} p_{00}^{21} \right\}$$

Then, $\langle p_1 \rangle = \frac{\sqrt{4\pi}}{3} a_{10}$

$\langle p_3 \rangle = \frac{\sqrt{4\pi}}{7} a_{30}$

- $\gamma_L^*$ has 0 helicity
- $s$-channel helicity conservation $\Rightarrow \pi^+\pi^- $ has 0 helicity
- Only $p_{00}$ states populated by $\gamma_L^*$

$$\langle p_1 \rangle + \frac{7}{3} \langle p_3 \rangle = \frac{4 + 2\sqrt{3}}{\sqrt{15}} p_{00}^{21} + \frac{2}{\sqrt{3}} p_{00}^{10}$$
Legendre Polynomials

\[
\frac{d\sigma}{d\cos(\theta)} = \sum_{l l'} a_{l l'} P_l(\cos(\theta)) P_{l'}(\cos(\theta))
\]

\[
P_0(x) = 1
\]

\[
P_1(x) = x
\]

\[
P_2(x) = \frac{1}{2} (3x^2 - 1)
\]

\[
P_3(x) = \frac{1}{2} (5x^3 - 3x)
\]
Fig. 2. The shape of two-pion mass $m_{\pi\pi}$ (GeV) distributions for pions with isospin one (dashed curve) and isospin zero (solid curves). The isospin zero distributions are plotted for Bjorken $x = 0.3, 0.4, 0.5$. (The larger $x_{Bj}$ the more enhanced is the distribution.)

Fig. 4. The ratio of $\frac{d^2\sigma}{dx^2} |_{I=0}$ to $\frac{d^2\sigma}{dx^2} |_{I=1}$ integrated over $m_{\pi\pi}$ from the threshold to 1.4 GeV as a function of Bjorken $x$. For $\frac{\sigma(I=0)}{\sigma(I=1)}$ at $t_{\text{min}}$
FIG. 6. \( \langle P_1(\cos \theta) \rangle \pi^n + (7/3) \langle P_3(\cos \theta) \rangle \pi^n \) as a function of \( m_{WW} \) with cross sections integrated over \( x_{Bj} \) from 0.05 to 0.4.

FIG. 7. \( \langle P_1(\cos \theta) \rangle \pi^n + (7/3) \langle P_3(\cos \theta) \rangle \pi^n \) as a function of \( x_{Bj} \) with cross sections integrated over \( m_{WW} \) from the threshold to 0.6 GeV. The dotted line shows the corresponding result obtained using the fit of [26] instead of the Padé approximation for the Omnès function \( f_3(m_{WW}) \).

FIG. 4. \( \langle P_1(\cos \theta) \rangle \pi^n \) as a function of \( x_{Bj} \) and \( m_{WW} \).

FIG. 5. \( \langle P_3(\cos \theta) \rangle \pi^n \) as a function of \( x_{Bj} \) and \( m_{WW} \).

Plots of 2-dv moments \( \langle P_3 \rangle \)

\( \langle x_{Bj}, M_{WW} \rangle \)
Deuterium $0.6 < m_{\pi\pi} < 0.95$ GeV

\[ \Delta E = \frac{M_X^2 - M^2}{2M} \]

**Signal**

**Counts**

**Counts**

\[ \cos \theta \]

- **x**: $\theta$ is $z$-axis
- *****: $N$ recoil is $-z$-axis (our choice)
Figure 4. \( m_{\pi\pi} \)-dependence of the intensity densities \( \langle P_1(\cos \theta) \rangle \), upper panels, and \( \langle P_3(\cos \theta) \rangle \), bottom panels, for both hydrogen and deuterium, left and right panels respectively. In the upper panels, the region \( 0.8 < m_{\pi\pi} < 1.1 \) GeV rebinned in finer channels to better investigate possible contributions from the narrow \( f_0(980) \) meson resonance. Also shown are leading twist predictions for the hydrogen target including the two-gluon exchange mechanism contribution, LPSG [4,5] (solid curve at \( x = 0.16 \)). A calculation without the gluon exchange contribution is showed for limited \( m_{\pi\pi} \) values, LPSG [6] (open squares at \( x = 0.1 \), open triangles at \( x = 0.2 \)). Fig. 1-a. In the above predictions, the contribution from \( f_0 \) meson decay was not considered. Instead, in the zoomed panel for the hydrogen target, the prediction from \([18]\), which includes the \( f_0 \) meson contribution, is shown. All experimental data have \( <x> = 0.16 \) and \( <Q^2> = 3 \) GeV\(^2\). The systematic uncertainty is represented by error band.

Figure 6. The \( z \)-dependence of the intensity densities \( \langle P_1(\cos \theta) \rangle \) for both targets separately, in the regions \( 0.30 < m_{\pi\pi} < 0.60 \) GeV (left panels) and \( 0.60 < m_{\pi\pi} < 0.95 \) GeV (right panels). Theoretical predictions from LPSG [6] (stars) for hydrogen are compared with the data. In these computations, the two-gluon exchange mechanism contribution to the process is neglected. The systematic uncertainty is given by the error band.
Figure 4. \( m_{\pi \pi} \)-dependence of the combinations \( \langle P_1(\cos \theta) + \sqrt{(7/3)} \cdot P_3(\cos \theta) \rangle \), upper panels, and \( \langle P_1(\cos \theta) - 14/9 \cdot P_3(\cos \theta) \rangle \), bottom panels, for both hydrogen and deuterium targets, left and right panels respectively. All experimental data have \( <z> = 0.16 \) and \( <Q^2> = 3 \text{ GeV}^2 \). The systematic uncertainty is given as error bands.
Experimental Moments
\[ \langle P_e \rangle \]

\[ \langle P_e \rangle = \frac{\int P_e(x) \sigma(x) \, dx}{\int \sigma(x) \, dx} \quad x = \cos \theta \]

**Method 1:** make histogram; numerically integrate

![Graph showing \( \sigma \) and \( P_e \sigma \) functions]

**Method 2:**

\[ \langle P_e \rangle = \frac{1}{N} \sum_j P_e(\cos \theta_j) \]

**Proof:** numerical integration

\[ \langle P_e \rangle = \frac{\sum_j P_e(x_j) N_j \Delta x}{\sum_j N_j \Delta x} \]

\[ N_j \equiv \# \text{ of counts in bin } j \]

\[ \langle P_e \rangle = \frac{1}{N} \sum_j P_e(x_j) N_j = \frac{1}{N} \sum_i P_e(x_i) \]

**Error:** Central Limit Theorem (standard error of \( \langle P_e \rangle \))

Variate \( X = \frac{1}{N} \sum_i x_i \) is normally distributed with \( \mu_X = \mu_x \) and \( \sigma_X = \sigma_x / \sqrt{N} \) for large \( N \).

**Advantage**

<table>
<thead>
<tr>
<th>Method 1</th>
<th>Can correct spectrum with MC</th>
<th>requires good statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method 2</td>
<td>easy</td>
<td>OK with limited statistics</td>
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</table>
Simple Monte Carlo

\[ \begin{align*}
\gamma^u &= (\nu, \gamma \sin \theta \cos \phi, \gamma \sin \theta \sin \phi, \gamma \cos \theta) \\
\theta &= 2\sin \sqrt{\frac{q^2}{4E'E'}} \\
\phi &= \sqrt{E'^2 - 2E'\cos \theta + E^2} \\
\phi &= \arcsin\left(\frac{E'\sin \theta}{q}\right) \\
\phi &= \text{random} \ [0, 2\pi] \\
\phi^\perp &= \text{random} \ [0, 2\pi] \\
\hat{P}_\perp &= p_\perp \left( \cos \phi^\perp \hat{\phi} + \sin \phi^\perp \hat{\theta} \right) \\
\hat{P}_\parallel &= p_\parallel \hat{\theta} \\
\hat{P}_\parallel &= \gamma \hat{\phi} \\
\hat{P} &= (\hat{P}_\parallel + \hat{P}_\perp)/\gamma \\
\gamma + \nu &= \sqrt{m_{\text{min}}^2 + p^2} + \sqrt{M^2 + r^2} \\
\gamma_\parallel + p_\parallel &= \gamma \\
M + \nu &= \sqrt{m_{\text{min}}^2 + p^2} + \sqrt{M^2 + r^2}
\end{align*} \]
Lorentz Boost

\[ \gamma = \frac{\sqrt{M_{\pi\pi}^2 + p^2}}{M_{\pi\pi}} \]

\( \Theta_{cm} \) chosen from specified distribution
\( \Phi_{cm} \) random \([0, 2\pi]\)

\[ \vec{P}_{\pm ll} = \gamma (\pm \vec{P}_\pi \cos \Theta_{cm} + \vec{p} \sqrt{M_{\pi\pi}^2 + p^2}) \hat{P} \]
\[ \vec{P}_{\pm \perp} = \pm \vec{P}_\pi \sin \Theta_{cm} [\cos \Phi_{cm} \hat{\Theta}_{cm} + \sin \Phi_{cm} \hat{\Phi}_{cm}] \]

Acceptance
\[ \Theta_x^{\pm} = \text{atan2}(p_x^{\pm}, p_z^{\pm}) \]
\[ \Theta_y^{\pm} = \text{atan2}(p_y^{\pm}, p_z^{\pm}) \]

\( P_\pm > 1 \text{ GeV} \)
\[-0.170 < \Theta_x^{\pm} < 0.170 \]
\[-0.140 < \Theta_y^{\pm} < -0.040 \]
\[0.040 < \Theta_y^{\pm} < 0.140 \]

if true output \( \cos \Theta_{cm} \)
Random Number Generation

\[ f(x) \, dx \] is the probability of finding \( x \) in the interval \( dx \).

\[ F(x) = \int_{-\infty}^{x} f(x) \, dx \]

\[ \Rightarrow \frac{dF}{dx} = f(x) \]

\[ dF = f(x) \, dx \]

Picking \( F \) at random and calculating \( x = F^{-1}(x) \) gives \( x \) with probability distribution \( f(x) \).

\[ f(x) = \frac{1}{2} (x+1) \text{ on } [-1,1] \]

\[ P_0, P_1 \]

\[ F(x) = \int_{-1}^{x} \frac{1}{2} (x+1) \, dx = \frac{1}{4} [x^2 + 2x + 1] \]

\[ x = 2F - 1 \]

\[ f(x) = \frac{1}{2} (1 + x^3) \text{ on } [-1,1] \]

\[ P_0, P_1, P_3 \]

\[ x = 2(F - \frac{3}{8}) / (1 + \frac{x^3}{4}) \text{ iterate} \]
Do isotropic CM distributions generate non-zero $\langle p_{T} \rangle$ and $\langle p_{z} \rangle$ due to holes in acceptance?

$\langle p_{z} \rangle$

NO!

$\langle p_{T} \rangle$

efficiency for $\pi^{-}$ detection in HERMES
OK down to about 50% efficiency

Expected $\langle p_3 \rangle$
0 6 bins in $x$ for $0.3 < m_{\pi\pi} < 0.6 \text{ GeV}$

□ 6 bins in $x$ for $0.6 < m_{\pi\pi} < 0.95 \text{ GeV}$

△ 11 bins in $m_{\pi\pi}$ for $x > 0.1$

INPUT $\langle x \rangle$, $\langle Q^2 \rangle$, $\langle m_{\pi\pi} \rangle$, $\langle p_T \rangle$ for each bin

In general, extracted $\langle p_T \rangle$ is far from expected value

Full averages over a bin will likely come closer to ideal value... but not completely,
\begin{align*}
\langle x \rangle &= 0.17 \\
\langle Q^2 \rangle &= 3.2 \text{ GeV}^2 \\
\langle m_{\pi^+} \rangle &= 0.52 \text{ GeV}
\end{align*}

Efficiency at average kinematic variables

$P_T$

$P_T$ distribution for event sample (varies little over $m_{\pi^+}$)
Green: with 250% efficiency cuts
Red: all data

\[ \langle p_7 \rangle \]

\[ \langle p_7 \rangle \]

\[ \langle p_7 \rangle \]

\[ \langle p_7 \rangle \]
$m_{\text{TOT}} = 0.5$
$\Delta \rho_7$
$\Delta \rho_1$
$x = 0.22$
$x = 0.12$
$x = 0.07$
Conclusions

• Toy Monte Carlo is a nice way to get an understanding of \( <P_n> \) within HERMES acceptance.

• Present estimations of errors in \( <P_n> \) due to acceptance are small compared to the statistical error bars.

↓

• Exclusive \( \pi^+\pi^- \) analysis and paper are OK as they presently stand.

• Any future analysis with improved statistics will need to reckon with acceptance corrections to \( <P_n> \)