Collective transport properties of bilayer-quantum-Hall excitonic condensates

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Double-layer electron systems in the quantum Hall regime have excitonic condensate ground states when the layers are close together and the total Landau level filling factor is close to an odd integer. In this paper we discuss the microscopic physics of these states and recent progress toward building a theory of the anomalous collective transport effects that occur in separately contacted bilayers when the condensate is present.

I. INTRODUCTION

Among the broken symmetry states that occur in many-particle systems, those in which long range phase coherence is established, either for bosons [1] or for pairs of fermions [2], have a special significance because of the quantum nature of their macroscopic order and the sometimes startling phenomenology that results. In semiconductors, the possibility of long-range phase coherence due to Bose condensation of excitons was first raised [3] nearly 40 years ago. In this paper, we argue that the anomalous transport properties discovered in bilayer quantum Hall systems by J.P. Eisenstein and collaborators [4] and studied in this group over the past couple of years, are an experimental manifestation of such an ordered state. We also briefly discuss the microscopic physics that causes this order to occur in the quantum Hall regime and efforts, currently in progress, to achieve a more complete understanding [5] of the collective transport effects that lie behind the experimental observations. This paper focuses on work related to these questions performed at the University of Texas.

II. SPONTANEOUS PHASE COHERENCE *IS* EXCITONIC BOSE CONDENSATION

Excitonic Bose condensation in semiconductors has usually been discussed by describing the fermionic degrees of freedom of a valence band in terms of missing electrons, holes, and the fermionic degrees of freedom of a conduction band directly in terms of electronic states. Under this particle-hole transformation a valence-band electron creation operator ($c^{\dagger}_{v,\vec{k}}$) is mapped to a hole annihilation operator ($d_{v,-\vec{k}}$). Here we have, for simplicity, dropped the spin degree-of-freedom. The subscript v is a band label that specifies the valence band; theoretical discussions of excitonic Bose condensation often focus on models with a single parabolic valence band. In electron-hole language, the phase coherence of electron-hole pairs in an excitonic condensate state proposed by Keldysh and Koslov [3] is signaled by a finite value for the expectation value $\langle c^{\dagger}_{c,\vec{k}}\, d^{\dagger}_{v,-\vec{k}}\rangle$. This quantity is closely analogous to the finite expectation value in superconductors for the operator which creates a pair of electrons in time-reversed states.

Two main difficulties have caused the experimental realization of this idea to be a challenge. Firstly, the ground state of a semiconductor has no electrons or holes present; it is necessary to generate the electron-hole plasma optically. This doesn't solve the problem completely however, since electrons and holes can also recombine optically; the electron-hole plasma that is created is always in a somewhat complex non-equilibrium state. Additionally, even if electrons and holes are present and optical recombination does not preempt their condensation, differences between conduction and valence band dispersions may be [6] an obstacle. The problems presented by electron-hole recombination can be mitigated to a large degree by placing electrons and holes in separate quantum wells. This idea was first suggested some time ago [7], and recent experiments on optically generated electron-hole plasmas in such systems have uncovered some interesting unundestood properties [8]. To date, however, there has not been any compelling evidence that a phase coherent condensate has been achieved in the absence of magnetic field. We will argue below that over the past couple of years excitonic pair condensation has however been achieved in the presence of an external magnetic field, although it has not generally been recognized as such.

The lack of recognition is, in our view, mainly a confusion of language. The situation is most clear if we consistently stick with most physically transparent description, instead of making a particle-hole transformation for the valence band, in describing the ordered state. The order parameter of the excitonic condensate state is then $\langle c^{\dagger}_{c,\vec{k}}\, c_{v,\vec{k}}\rangle$. Excitonic condensation is nothing but the development of spontaneous phase coherence between two different bands in a solid, spontaneous coherence in the case of bilayers between electron bands localized in two different quantum wells. This is precisely what occurs for bilayer electron systems in the quantum Hall regime near odd integer total filling factors. The prediction that this broken symmetry would occur in bilayer quantum Hall systems [9–11], early experimental indications [10,12] of coherence-dependent Hall gaps, and many aspects of the theory of these states [13] were, however, developed largely [14] without reference to the literature on excitonic Bose condensation. Because their spontaneous coherence occurs between two different two-dimensional *conduction subbands*, there is no need to simultaneously populate the conduction band with electrons and the valence band with holes in order to establish conditions under which the broken symmetry can occur. This eliminates one key difficulty that has plagued efforts to realize excitonic condensation. At the same time dispersionless Landau bands eliminate the requirement of Fermi surface nesting [6].

III. WHY EXCITONIC CONDENSATION BETWEEN CONDUCTION BANDS IS LIKELY ONLY IN THE QUANTUM HALL REGIME

In this section, we use mean-field theory to explain why excitonic condensation (spontaneous coherence) at zero magnetic field is more likely between quasi-2D subbands that have opposite energy-wavevector dispersions, *i.e.* between conduction and valence subbands, and why this tendency no longer applies in the strong magnetic field limit. In mean-field theory, the ordered states we are interested in are Slater determinants formed from quasiparticles with spontaneous coherence and are closely analogous to the BCS variational wavefunctions familiar from the theory [2] of superconductivity. Thus

$$|\Psi\rangle = \prod_{\vec{k}\in S}[u_{\vec{k}}\, c^{\dagger}_{T,\vec{k}} + v_{\vec{k}}\, c^{\dagger}_{B,\vec{k}}] \prod_{\vec{k}\in D} c^{\dagger}_{T,\vec{k}}\, c^{\dagger}_{B,\vec{k}}|0\rangle \tag{1}$$

where $\vec{k}$ is a Bloch wavevector, $|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$, $\vec{k} \in S$ and $\vec{k} \in D$ are exclusive sets of singly and doubly occupied wavevectors, and the phase relationship between $u_{\vec{k}}$ and $v_{\vec{k}}$ is fixed. Here the subscript T labels *top* layer electrons while B labels *bottom* layer electrons. This form of variational wavefunction is more general than the simple BCS variational wavefunction in that allows some wavevectors $\vec{k} \in D$ to be occupied by a pair of quasiparticles; these wavevectors do not contribute to spontaneous coherence. Unlike a Fermi gas, which has all wavevectors inside the Fermi sea occupied in both layers, top and bottom layer electrons in this variational wavefunction are correlated and have a reduced probability of being close together. The fact that these correlations are present can be verified by an explicit calculation or understood in terms of the fact that two electrons cannot have the same wavevector $\vec{k} \in S$ whether they are in the same layer or opposite layers. This state has spontaneous phase coherence when the set $\vec{k} \in S$ is not empty. Spontaneous coherence is established in order to improve interlayer correlations, but comes at a cost in intralayer correlations and kinetic energy.

The dependence of this competition on band dispersion can be understood qualitatively from the following considerations. The cost in kinetic energy tends to be minimized when there is a set of wavevectors for which single occupation occurs even in the non-interacting electron ground state. For example, for parabolic bands and equal densities of conduction band electrons and valence band holes, one state is occupied at each Bloch wavevector in the non-interacting ground state. When the two bands have the same dispersion, however, either the two bands have a large energetic offset which works against order or the kinetic energy is minimized by occupying Bloch states for both bands at most wavevectors. This is not to say that spontaneous coherence between Bloch bands in a solid that have the same curvature is impossible; in fact ferromagnetism can be regarded as being due to spontaneous coherence between bands that are distinguished only by their spin index. (This view is natural when the spin-quantization axis is chosen to be in the plane perpendicular to the spontaneous magnetization direction.) Whenever the conditions are ripe for this type of order, however, it will usually be favorable to to establish coherence between bands with the same orbital label rather than different ones; ferromagnetism preempts spontaneous coherence between two conduction bands. In the quantum well case we know that ferromagnetism occurs within a single layer only at unattainably low electron densities, if at all. Spontaneous coherence between two-dimensional conduction subbands localized in different layers is therefore extremely unlikely. Spontaneous coherence between conduction and valence subbands in different layers is much more likely, perhaps even probable, but hard to realize in experiment. All this changes in the quantum Hall regime.

We now turn our attention to a bilayer quantum Hall system with total filling factor $\nu = 1$, the most favorable circumstance for spontaneous interlayer phase coherence. For simplicity we assume that the magnetic field is strong enough that all electrons in both layers are in the lowest kinetic energy, majority spin, Landau level. The mean-field variational wavefunction for a phase coherent bilayer state has the form:

$$|\Psi\rangle = \prod_k [u_k c^\dagger_{T,k} + v_k c^\dagger_{B,k}]|0\rangle \qquad (2)$$

where $|u_k|^2 + |v_k|^2 = 1$ and k is the one-component wavevector that labels states in the lowest Landau level in Landau gauge. In this case it turns out that, partly because of the degeneracy of the Landau bands, u and v are independent of k in the translational invariant mean-field ground state. Since there is one occupied orbital for each k, this state indeed has Landau level filling factor $\nu = 1$. The mean-field state for the case of a conduction band Landau level in one layer and a valence band Landau level in the other layer would be identical, with u and v independent of k in the ground state and no Landau gauge orbitals either empty or doubly occupied. Since there is no dispersion of the Landau band in either case, there is no reason to expect any qualitative dependence on the microscopic band from which the Landau level is derived. This important property was apparently not appreciated in the literature [15] on excitonic condensation in strong fields, likely because it is exclusively couched in the language of conduction band electrons and valence band holes. (For $\nu \neq 1$, some states in the variational wavefunction must be either empty or doubly occupied, weakening the tendency toward order as discussed above. The physics of this suppression is, we believe, quite sensitive to the disorder present in the quantum wells.) When the two subband energies are identical, the mean-field state minimizes the kinetic energy for any choice of its variational parameters, u_k and v_k, and also, by establishing interlayer phase coherence, reduces the interlayer interaction energy. It turns out [16] that this mean-field wavefunction with $u_k = v_k = \exp(i\phi)/\sqrt{2}$ and arbitrary ϕ approaches the exact ground state of $\nu = 1$ bilayer quantum Hall systems for layer spacing $d \to 0$. Corrections to mean-field-theory can [17] be calculated perturbatively for small d, but for large d, quantum fluctuation corrections to the mean-field state become substantial and eventually a phase transition occurs to a state without spontaneous coherence.

IV. COLLECTIVE TRANSPORT PROPERTIES

Bilayer quantum Hall ferromagnets have most often been described as pseudospin ferromagnets. If states in the top layer are regarded as having pseudospin up and states in the bottom layer are regarded as having pseudospin down, the coherence factors for a pseudospin with orientation specified by polar and azimuthal angles θ and ϕ are given by the familiar spin-1/2 coherent state expressions: $u_k = \cos(\theta/2)$ and $v_k = \sin(\theta/2)\exp(i\phi)$. Why, then, have we been describing them as excitonic condensates, which are expected to have superfluid properties? The answer to this question appears to be [18] that ferromagnets can in principle exhibit superfluid behaviors similar to those of exitonic condensates, provided that they have nearly perfect easy-plane magnetic anisotropy. This kind of spintronic effect, in which the order parameter field is driven to a metastable configuration, is qualitatively different from giant magnetoresistance, current induced magnetization reversal, or any of the other interesting effects [19] that have been studied in ferromagnetic metals over the past decade. Advances in metal spintronics do however bring us closer to being able to realize the conditions required for their observation. In this section we switch from excitonic to magnetic language in briefly discussing one collective transport anomaly that has been seen in phase coherent bilayer quantum Hall systems. Our intention in doing so is to suggest that the transport anomalies very similar to those recently observed in quantum Hall bilayers, could also occur in thin film ferromagnets. Similar anomalies should also occur in zero field excitonic condensates when phase coherence is eventually achieved there; something that we await with confidence.

We are interested in modeling two-terminal I-V measurements in which a voltage difference is applied between contacts that are on opposite ends of the bilayer system and make contact only with top and bottom layers respectively. What was discovered experimentally [4] is that a large zero-bias peak emerges in the two-terminal conductance when interlayer phase coherence is established. In this paper, we discuss a purely one-dimensional model that, because of the importance of edge states in the quantum Hall regime, is not adequate to fully describe the experimental situation; work on a fully two-dimensional models that account for $\mathbf{E} \times \mathbf{B}$ drift near the edge of the sample is currently in progress. Despite its relative simplicity some ideas that we believe to be key emerge from the analysis described below. We describe collective transport [4] using equations of motion for the order parameter field that can be derived microscopically [20], at least in the linear response regime. These equations are basically the Landau-Liftshitz equations that we expect for systems with magnetic order, but some features are special in the quantum Hall regime. We describe the order parameter field at a given position and time in terms of two variables ϕ, the azimuthal angle of the pseudospin field, and its direction cosine along the polar direction Ω_z. More physically, ϕ describes the relative phase between condensed electrons in the two layers while Ω_z describes the degree of charge transfer between the layers. In the excitonic condensate language ϕ is the phase of the electron-hole Cooper pair field. In the present analysis we assume that Ω_z is always small, *i.e.* that the fraction of the local charge density transferred from one layer to another is always small, but we allow ϕ to vary arbitrarily.

We have derived [20,21] the following equations of motion for the order parameter field,

$$\frac{\partial\varphi}{\partial t} = -\frac{4\pi l^2\beta}{\hbar}\Omega_z - \frac{\alpha_\varphi}{\hbar}\left[\frac{\Delta_t}{2}\sin\varphi - 2\pi l^2\rho_s\nabla^2\varphi\right] \tag{3}$$

$$\frac{\partial\Omega_z}{\partial t} = \frac{1}{\hbar}\left[\frac{\Delta_t}{2}\sin\varphi - 2\pi l^2\rho_s\nabla^2\varphi\right] + \frac{8\pi^2 l^4\beta\sigma_z}{e^2 M_0}\nabla^2\Omega_z \tag{4}$$

where [16] ρ_s is the pseudospin stiffness, β is a measure of the capacitive energy cost of tilting the pseudospin out if its x-y easy plane, Δ_t is the interlayer tunneling amplitude, M_0 is the order parameter amplitude, α_φ is a dimensionless Gilbert damping parameter for the pseudospin field, and σ_z is a transverse pseudospin conductivity. In these equations $4\pi\ell^2\beta\Omega_z$ is the electrochemical potential difference between top and bottom layers, the $\hat{z}$ direction component of the pseudospin effective magnetic field. The first term on the right hand side of Eq. 3 describes precession of the collective pseudospin order parameter field around the z direction, while the second term describes damping processes which allow the x-y component of the pseudospin to align with the x-y component of the effective pseudospin field. The damping term is non-zero when this alignment is imperfect. Since inter-layer tunneling favors equal phases in the two-layers, it contributes a pseudospin-effective field in the $\hat{x}$ direction to the Hamiltonian; $\Delta_t \sin(\varphi)/2$ is the component of the pseudospin field which is out of alignment with a pseudospin that has azimuthal orientation φ. The second term arises from the microscopic exchange fields which produce pseudospin effective fields [22] that are oriented along a direction obtained by averaging over spatially nearby pseudospin orientations. When the pseudospin orientation varies spatially, this average will not be locally aligned with the pseudospin direction.

The second equation describes the precession of the $\hat{z}$ component of the pseudospin field around the transverse x-y plane field, and damping terms that allow this component of the pseudospin field to decay toward alignment with the effective magnetic field. An interesting aspect of the physics of these bilayer quantum Hall ferromagnets is that the Gilbert damping coefficient in this equation vanishes [20]. Since decay of the pseudospin current cannot occur locally, we include in our analysis a quasiparticle transport term, which allows the $\hat{z}$ component of the pseudospin field to decay when microscopic pseudospin-polarized currents have finite divergence. The three terms on the right hand side of Eq. 4 can be identified as representing collective inter-layer tunneling, two-dimensional supercurrent, and quasiparticle current contributions to $\partial\Omega_z/\partial t$.

We believe that these Landau-Liftshitz equations describe the essence of order parameter dynamics in quantum Hall bilayer coherent states. In order to build a theory of two-terminal inter-layer transport I-V curves, we must prescribe boundary conditions at the edge of the system. We focus here on balanced bilayers for which charge (sum of current in the two layers) and pseudospin polarized (difference of current in the two layers) currents can be considered separately. In the inter-layer tunneling geometry experiment we focus on the pseudospin-current which changes sign (in the top layer and out the bottom) across the sample. Since the two-terminal voltage is measured between a top layer contact on one side of the sample and a bottom layer contact on the other side of the sample, its pseudospin-contribution will be the sum of chemical potential differences between top and bottom layers on the two ends of the sample, which will identical by symmetry.

$$\mu_z|_{-L/2} = \frac{eV_{PS}}{2} \tag{5}$$

$$\mu_z|_{L/2} = \frac{eV_{PS}}{2} \tag{6}$$

where $\mu_z = 4\pi\ell^2\beta\Omega_z$, V_{PS} is the pseudospin contribution to the measured voltage difference, and L is the linear size of the sample. The total measured voltage also has a contribution from the charge current channel, but this will be much smaller; in the following we regard V_{PS} as the total two-terminal voltage V.

The large zero-bias peak in the two-terminal conductance can be understood in terms of stationary solutions of these equations for collective dynamics. We find that the time-independent pseudospin fields satisfy

$$\frac{\mathrm{d}^2\Omega_z}{\mathrm{d}x^2} - \frac{1}{L_z^2}\Omega_z = 0 \tag{7}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} - \frac{1}{\lambda_j^2}\sin\varphi = \frac{2\beta}{\alpha_\varphi \rho_s}\Omega_z \tag{8}$$

where we have introduced the two length scales defined by these equations. λ_j is a domain wall width in the magnetic language and a *Josephson length* in the excitonic superfluid language. It is defined by

$$\lambda_j = l\sqrt{|\frac{4\pi\rho_s}{\Delta_t}}.$$

The length L_z is discussed below. These equations can be solved using the boundary condition that $d\varphi/dx$ vanish that both ends of the sample, *i.e.* that no supercurrent flow into or out of the sample. This leads to

$$\Omega_z = \frac{eV}{8\pi l^2\beta\cosh\left(\frac{L}{2L_z}\right)}\cosh\left(\frac{x}{L_z}\right) \tag{9}$$

where the length L_z is defined by:

$$L_z = l\sqrt{\frac{2\pi\hbar\sigma_z\alpha_\varphi}{e^2 M_0\left(\alpha_z + \alpha_z\alpha_\varphi\right)}}$$

L_z is the length scale for relaxation of the pseudospin polarization of the quasiparticle current injected from the contacts. In the limit $L_z \ll \lambda_j$, the stationary solution to the damped Landau-Liftshitz-Gilbert equations reflects a process in which the injected quasiparticle pseudospin current is converted into a supercurrent, which then decays by collective interlayer tunneling. These solutions exist only up to a critical value of the source-drain voltage V, explaining the sharp peak in the two-terminal conductivity near zero bias. $I(V)$ can be evaluated from the following expression for the purely quasiparticle currents entering and exiting the sample:

$$j = \frac{4\pi l^2\beta\sigma_z}{e}\left|\frac{\partial\Omega_z}{\partial x}\right|_{\pm L/2}$$

It follows that the two-terminal resistance of this theory is proportional to the product of the quasiparticle resistivity and the length L_z over which the quasiparticle current is present. Beyond the critical voltage only time-dependent solutions to these equations exist and the time averaged current decreases.

V. SUMMARY

Bilayer quantum Hall systems near total filling factor $\nu = 1$ have broken symmetry ground states with spontaneous interlayer phase coherence between two-dimensional subbands that are localized in different layers. Spontaneous coherence between different bands is the property that defines excitonic Bose condensation, leading for example to superfluid behavior for currents flowing in opposite directions in the two bands. The recent observation of anomalous collective transport effects in the quantum Hall regime when bilayer coherence is present therefore represents the experimental discovery of the long-sought excitonic Bose-Einstein condensate. In this paper we have briefly discussed the possibility of similar superfluid collective transport effects in thin-film ferromagnets with nearly-perfect easy-plane anisotropy and efforts to develop a complete picture of two-terminal bilayer quantum Hall transport measurements in which current is injected into one of the electron layers and removed from the other.

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